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If $a, \mathrm{b}, \mathrm{c} > 0, a + \mathrm{b} + \mathrm{c} = 3$ and $\lambda \geq 0$ then :

$$\sum_{\text{cyc}} \frac{a^2 + a\mathrm{b}^2}{\mathrm{b}^2 + a + \lambda\mathrm{b}} \geq \frac{6}{\lambda + 2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}} \frac{a^2 + a\mathrm{b}^2}{\mathrm{b}^2 + a + \lambda\mathrm{b}} = \sum_{\text{cyc}} \frac{a(a + \mathrm{b}^2 + \lambda\mathrm{b} - \lambda\mathrm{b})}{\mathrm{b}^2 + a + \lambda\mathrm{b}} = \sum_{\text{cyc}} a - \sum_{\text{cyc}} \frac{\lambda a\mathrm{b}}{\mathrm{b}^2 + a + \lambda\mathrm{b}}$$

$$\overset{\text{AM-GM}}{\geq} \sum_{\text{cyc}} a - \sum_{\text{cyc}} \frac{\lambda a\mathrm{b}}{2\mathrm{b}.\sqrt{a} + \lambda\mathrm{b}} = \sum_{\text{cyc}} \left(a - \frac{\lambda a}{2\sqrt{a} + \lambda}\right) = 2\sum_{\text{cyc}} \frac{a.\sqrt{a}}{2\sqrt{a} + \lambda}$$

$$= 2\sum_{\text{cyc}} \frac{a^2}{2a + \lambda.\sqrt{a}} \overset{\text{Bergstrom}}{\geq} \frac{2\left(\sum_{\text{cyc}} a\right)^2}{2\sum_{\text{cyc}} a + \lambda.\sum_{\text{cyc}} \sqrt{a}} \overset{\text{CBS}}{\geq} \frac{2\left(\sum_{\text{cyc}} a\right)^2}{2\sum_{\text{cyc}} a + \lambda.\sqrt{3\sum_{\text{cyc}} a}}$$

$$\overset{a+\mathrm{b}+\mathrm{c}=3}{=} \frac{2.9}{2.3 + 3\lambda} = \frac{6}{\lambda + 2} \therefore \sum_{\text{cyc}} \frac{a^2 + a\mathrm{b}^2}{\mathrm{b}^2 + a + \lambda\mathrm{b}} \geq \frac{6}{\lambda + 2}$$

$$\forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\middle|\, a + \mathrm{b} + \mathrm{c} = 3 \text{ and } \lambda \geq 0, '' ='' \text{ iff } a = \mathrm{b} = \mathrm{c} = 1 \text{ (QED)}$$