ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 1$ *and* $0 \le \lambda \le 3$, then :

$$(\lambda+1)\left(\sum_{cyc} xy\right)^2 \le \frac{2\lambda+3}{9}\left(\sum_{cyc} xy\right) + \lambda xyz$$

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$$(\lambda+1)\left(\sum_{cyc} xy\right)^2 \le \frac{2\lambda+3}{9}\left(\sum_{cyc} xy\right) + \lambda xyz \overset{x+y+z=1}{\Leftrightarrow} (\lambda+1)\left(\sum_{cyc} xy\right)^2$$

$$\le \frac{2\lambda+3}{9}\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 + \lambda xyz\sum_{cyc} x$$

$$\Leftrightarrow \lambda\left(9\left(\sum_{cyc} xy\right)^2 - 2\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 - 9xyz\sum_{cyc} x\right) \le$$

$$3\left(\sum_{cyc} xy\right)\left(\left(\sum_{cyc} x\right)^2 - 3\sum_{cyc} xy\right) \text{ and it's trivially true when :}$$

$$9\left(\sum_{cyc} xy\right)^2 - 2\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 - 9xyz\sum_{cyc} x < 0 \; (\because \lambda \ge 0) \;\&\; \text{when :}$$

$$9\left(\sum_{cyc} xy\right)^2 - 2\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 - 9xyz\sum_{cyc} x \ge 0, \text{then :}$$

$$\lambda\left(9\left(\sum_{cyc} xy\right)^2 - 2\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 - 9xyz\sum_{cyc} x\right) \overset{\lambda \le 3}{\le}$$

$$3\left(9\left(\sum_{cyc} xy\right)^2 - 2\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x\right)^2 - 9xyz\sum_{cyc} x\right)$$

$$\overset{?}{\le} 3\left(\sum_{cyc} xy\right)\left(\left(\sum_{cyc} x\right)^2 - 3\sum_{cyc} xy\right) \Leftrightarrow \sum_{cyc}(x^3y + xy^3) - 2\sum_{cyc} x^2y^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \sum_{cyc} xy(x-y)^2 \overset{?}{\geq} 0 \to \text{true} \therefore (\lambda+1)\left(\sum_{cyc} xy\right)^2 \leq \frac{2\lambda+3}{9}\left(\sum_{cyc} xy\right) + \lambda xyz$$

$$\forall\, x, y, z > 0 \,\Big|\, x+y+z=1 \text{ and } 0 \leq \lambda \leq 3, \text{"="} \text{ iff } x=y=z=\frac{1}{3} \text{ (QED)}$$