ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 3$ and $\lambda \geq 3$, then :

$$\sum_{cyc} \sqrt{\frac{yz}{\lambda - x}} \leq \frac{3}{\sqrt{\lambda - 1}}$$

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$$\text{Since } x, y, z > 0 \,|\, x + y + z = 3 \therefore x, y, z < 3 \leq \lambda \Rightarrow \lambda - x, \lambda - y, \lambda - z > 0$$

$$\text{and so, } \sum_{cyc} \sqrt{\frac{yz}{\lambda - x}} \overset{\text{AM-GM}}{\leq} \sum_{cyc} \frac{y + z}{2.\sqrt{\lambda - x}} \overset{x+y+z=3}{=} \sum_{cyc} \frac{3 - x}{2.\sqrt{\lambda - x}} \overset{\text{Jensen}}{\leq} \frac{3}{2} \cdot \frac{3 - \frac{x+y+z}{3}}{\sqrt{\lambda - \frac{x+y+z}{3}}} \overset{x+y+z=3}{=}$$

$$\frac{3}{\sqrt{\lambda - 1}} \left(\because f(t) = \frac{3 - t}{\sqrt{\lambda - t}} \; \forall\, t \in (0, 3) \Rightarrow f''(t) = \frac{t + 9 - 4\lambda}{4(\lambda - t)^{\frac{5}{2}}} \overset{t \in (0,3)}{<} \frac{4(3 - \lambda)}{4(\lambda - t)^{\frac{5}{2}}} \overset{\lambda \geq 3}{\leq} 0 \Rightarrow f(t) \text{ is concave} \right)$$

$$\therefore \sum_{cyc} \sqrt{\frac{yz}{\lambda - x}} \leq \frac{3}{\sqrt{\lambda - 1}} \; \forall\, x, y, z > 0 \,|\, x + y + z = 3 \text{ and } \lambda \geq 3,$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$