ROMANIAN MATHEMATICAL MAGAZINE

For $x, y, z > 0, k \geq \sqrt{2}$ prove that :

$$\left(k+1-k.\frac{x}{y}\right)^2+\left(k+1-k.\frac{y}{z}\right)^2+\left(k+1-k.\frac{z}{x}\right)^2\geq 3$$

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$$\left(k+1-k.\frac{x}{y}\right)^2+\left(k+1-k.\frac{y}{z}\right)^2+\left(k+1-k.\frac{z}{x}\right)^2\geq 3$$

$$\Leftrightarrow 3k^2+6k+k^2\sum_{cyc}\frac{x^2}{y^2}-2k(k+1)\sum_{cyc}\frac{x}{y}\geq 0$$

$$\Leftrightarrow 3k+6+k\sum_{cyc}a^6-2(k+1)\sum_{cyc}a^3\geq 0\ \left(\sqrt[3]{\frac{x}{y}}=a,\sqrt[3]{\frac{y}{z}}=b,\sqrt[3]{\frac{z}{x}}=c\right)$$

$$\Leftrightarrow (3k+6)a^2b^2c^2+k\sum_{cyc}a^6-2(k+1)abc\sum_{cyc}a^3\geq 0\ (\because abc=1)$$

$$\Leftrightarrow k\left(\sum_{cyc}a^6+3a^2b^2c^2-2abc\sum_{cyc}a^3\right)-2abc\left(\sum_{cyc}a^3-3abc\right)\geq 0$$

$$\Leftrightarrow k\left(\begin{array}{c}6a^2b^2c^2+\left(\sum_{cyc}a^2\right)\left(\left(\sum_{cyc}a^2\right)^2-3\sum_{cyc}a^2b^2\right)-\\ 2abc\left(3abc+\left(\sum_{cyc}a\right)\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)\right)\end{array}\right)$$

$$-2abc\left(\left(\sum_{cyc}a\right)\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)\right)\geq 0\Leftrightarrow$$

$$k\left((p^2-2q)^3-3(p^2-2q)(q^2-2rp)-2r(p^3-3pq)\right)-2r(p^3-3pq)\geq 0$$

$$\left(p=\sum_{cyc}a\,,q=\sum_{cyc}ab\,,r=abc\right)\Leftrightarrow$$

$$k\left((p^2-2q)^3-3q^2(p^2-2q)\right)+r(k(4p^3-6pq)-2p^3+6pq)\geq 0$$

$$\Leftrightarrow \mathbf{Ar^2 + r.G(p,q) + H(p,q)} \boxed{\overset{(*)}{\geq}} \mathbf{0}$$

$$\begin{pmatrix} \mathbf{A = 0}, \mathbf{G(p,q) = k(4p^3 - 6pq) + 2p^3 - 6pq,} \\ \mathbf{H(p,q) = k\left((p^2 - 2q)^3 - 3q^2(p^2 - 2q)\right)} \end{pmatrix}$$

$\boldsymbol{and} \because \mathbf{F(\boldsymbol{a},b,c) = Ar^2 + r.G(p,q) + H(p,q) \geq 0}$ **for** $\mathbf{A \leq 0 \leq \boldsymbol{a},b,c}$ **holds**
iff $\mathbf{F(\boldsymbol{a},1,1) \geq 0}$ ***and*** $\mathbf{F(0,b,c) \geq 0} \therefore (*)$ **is true iff** $\mathbf{F(\boldsymbol{a},1,1) \geq 0}$ ***and*** $\mathbf{F(0,b,c) \geq 0}$;

$$\text{where } \mathbf{F(\boldsymbol{a},b,c) = k\left(\sum_{cyc} \boldsymbol{a}^6 + 3\boldsymbol{a}^2b^2c^2 - 2\boldsymbol{a}bc\sum_{cyc} \boldsymbol{a}^3\right) - 2\boldsymbol{a}bc\left(\sum_{cyc} \boldsymbol{a}^3 - 3\boldsymbol{a}bc\right)}$$

Now, $\mathbf{F(0,b,c) = k(b^6 + c^6) > 0}$ ***and*** $\mathbf{F(\boldsymbol{a},1,1) =}$

$$\mathbf{k\left(\boldsymbol{a}^6 + 2 + 3\boldsymbol{a}^2 - 2\boldsymbol{a}(\boldsymbol{a}^3 + 2)\right) - 2\boldsymbol{a}(\boldsymbol{a}^3 + 2 - 3\boldsymbol{a})}$$

$$\mathbf{= k(\boldsymbol{a} - 1)^2.(\boldsymbol{a}^4 + 2\boldsymbol{a}^3 + \boldsymbol{a}^2 + 2) - 2\boldsymbol{a}(\boldsymbol{a} - 1)^2.(\boldsymbol{a} + 2) \overset{?}{\geq} 0}$$

$$\Leftrightarrow \mathbf{k^2(\boldsymbol{a}^4 + 2\boldsymbol{a}^3 + \boldsymbol{a}^2 + 2)^2 \overset{?}{\geq} 4\boldsymbol{a}^2(\boldsymbol{a} + 2)^2 \; (\because (\boldsymbol{a} - 1)^2 \geq 0) \text{ and } \because k^2 \geq 2}$$

$$\therefore \text{ it suffices to prove : } \mathbf{2(\boldsymbol{a}^4 + 2\boldsymbol{a}^3 + \boldsymbol{a}^2 + 2)^2 \overset{?}{\geq} 4\boldsymbol{a}^2(\boldsymbol{a} + 2)^2}$$

$$\Leftrightarrow \mathbf{2\left(\boldsymbol{a}^8 + 4\boldsymbol{a}^7 + 6\boldsymbol{a}^6 + 4\boldsymbol{a}^5 + 2\boldsymbol{a}^4 + (\boldsymbol{a}^2 - 2)^2\right) \overset{?}{\geq} 0 \rightarrow \text{true (strict)} \because \boldsymbol{a} > 0}$$

and **so,** $\mathbf{F(\boldsymbol{a},1,1) \geq 0}$ ***and*** $\mathbf{F(0,b,c) > 0 \Rightarrow F(\boldsymbol{a},b,c) \geq 0 \Rightarrow (*)}$ **is true**

$$\therefore \mathbf{\left(k + 1 - k.\frac{x}{y}\right)^2 + \left(k + 1 - k.\frac{y}{z}\right)^2 + \left(k + 1 - k.\frac{z}{x}\right)^2 \geq 3 \; \forall \, x,y,z > 0 \text{ and } k \geq \sqrt{2},}$$

$$\mathbf{'' ='' \text{ iff } x = y = z \text{ (QED)}}$$