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Let $a,b,c,m,n\geq 0$ and $abc=1$. Prove that

$$\frac{1}{a^{2m+n}+a^{2n+m}+1}+\frac{1}{b^{2m+n}+b^{2n+m}+1}+\frac{1}{c^{2m+n}+c^{2n+m}+1}\geq 1$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $a^{2m+n}+a^{2n+m}=a^{2(m+n)}+a^{m+n}-a^{m+n}(a^m-1)(a^n-1)\leq a^{2(m+n)}+a^{m+n}$,

because a^m-1, a^n-1 have the same sign.

And since $a^{m+n}.b^{m+n}.c^{m+n}=1$, then $\exists x,y,z>0$ such that

$$a^{m+n}=\frac{yz}{x^2}, b^{m+n}=\frac{zx}{y^2}, c^{m+n}=\frac{xy}{z^2}.$$

Therefore

$$\sum_{cyc}\frac{1}{a^{2m+n}+a^{2n+m}+1}\geq\sum_{cyc}\frac{1}{a^{2(m+n)}+a^{m+n}+1}=\sum_{cyc}\frac{x^4}{(yz)^2+x^2yz+x^4}$$

$$\overset{AM-GM}{\geq}\sum_{cyc}\frac{x^4}{(yz)^2+\frac{(xy)^2+(zx)^2}{2}+x^4}\overset{CBS}{\geq}\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}\left[(yz)^2+\frac{(xy)^2+(zx)^2}{2}+x^4\right]}=1.$$

So the proof is complete. Equality holds iff $a=b=c=1$.