PROBLEMS FOR JUNIORS

JP.571. If $x, y, z > 0, x + y + z = 3$ then find the minimum value of

$$P = \sum \frac{y+z}{x^3+2} + \frac{1}{2}\sum x^2$$

Proposed by Marin Chirciu - Romania

JP.572. If $a, b, c, d > 0$ then:

$$a\sqrt{\frac{a}{a^3+3bcd}} + b\sqrt{\frac{b}{b^3+3cda}} + c\sqrt{\frac{c}{c^3+3dab}} + d\sqrt{\frac{d}{d^3+3abc}} \geq 2$$

Proposed by Daniel Sitaru - Romania

JP.573. If $a, b, c > 0$ and $\lambda \geq \frac{1}{4}$ then:

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} + \lambda\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \geq \frac{3}{2}(2\lambda + 1)$$

Proposed by Marin Chirciu - Romania

JP.574. In ΔABC the following relationship holds:

$$\frac{27}{8} \leq \frac{(a+b+c)^3}{(a+b)(b+c)(c+a)} \geq \frac{27R}{16r}$$

Proposed by Marin Chirciu - Romania

JP.575. Find $x, y \in \mathbb{R}$ such that:

$$\sqrt{x^2+y^2} + \sqrt{(x-4)^2+(y-3)^2} + \sqrt{(x-4)^2+y^2} + \sqrt{x^2+(y-3)^2} = 10$$

Proposed by Daniel Sitaru - Romania

JP.576. If $a, b, c > 0, a + b + c = 3$ and $0 \leq \lambda \leq 3$ then:

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} + \frac{\lambda abc}{a^2b + b^2c + c^2a + abc} \geq \frac{\lambda + 6}{4}$$

Proposed by Marin Chirciu - Romania

JP.577. If $x, y, z > 0, xyz = 1$ and $n \in \mathbb{N}^*$ then:

$$\sum \frac{x^{2n+1}}{x+y+1} \geq 1$$

Proposed by Marin Chirciu - Romania

JP.578. If $x, y, z > 0$ and $n \in \mathbb{N}, n \geq 2$ in ΔABC:

$$\sum \frac{x^n a^{2n-1}}{(y+z)^n} \geq \frac{\sqrt{3}}{2r}(6r^2)^n$$

Proposed by Marin Chirciu - Romania

JP.579. Solve for real numbers:

$$\begin{cases} \frac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x-y)+yz(y-z)+zx(z-x)} = 55 \\ x^3+y^3+z^3 = 99 \\ \frac{xy(x^3-y^3)+yz(y^3-z^3)+zx(z^3-x^3)}{xy(x^2-y^2)+yz(y^2-z^2)+zx(z^2-x^2)} = \frac{55}{9} \end{cases}$$

Proposed by Daniel Sitaru - Romania

JP.580. If $a, b \geq 0; x \in (0, \frac{\pi}{2})$ then:

$$\frac{[2^a+\sin^2 x]+[2^b+\cos^2 x]}{[2^a+2^b]} + \frac{[2^b+\sin^2 x]+[2^c+\cos^2 x]}{[2^b+2^c]} +$$
$$+ \frac{[2^c+\sin^2 x]+[2^a+\cos^2 x]}{[2^c+2^a]} \geq 3$$

$[*]$ - great integer function.

Proposed by Daniel Sitaru - Romania

JP.581. If $x, y, z \geq 0$ then:

$$\frac{[3^x]\cdot[3^y]}{[3^{x+y}]} + \frac{[3^y]\cdot[3^z]}{[3^{y+z}]} + \frac{[3^z]\cdot[3^x]}{[3^{z+x}]} \leq 3$$

$[*]$ - great integer function.

Proposed by Daniel Sitaru - Romania

JP.582. In acute $\Delta ABC; AA', BB', CC'$ - altitudes; H - orthocenter; m_a, m_b, m_c - medians; r - inradii. Prove that:

$$\frac{m_a^2}{HA'} + \frac{m_b^2}{HB'} + \frac{m_c^2}{HC'} \geq \frac{81r^2(\tan A + \tan B + \tan C)}{h_a \tan A + h_b \tan B + h_c \tan C}$$

Proposed by Daniel Sitaru - Romania

JP.583. In $\Delta ABC, a, b, c$ - sides, h_a, h_b, h_c - altitudes, s - semiperimeter, the following relationship holds:

$$\frac{h_a^2}{b^2+c^2}+\frac{h_b^2}{c^2+a^2}+\frac{h_c^2}{a^2+b^2}\le\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Daniel Sitaru - Romania

JP.584. Prove that in any triangle ABC the following inequality holds:

$$\sum_{cyc}\cos^3\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ge\frac{3}{16}\sum_{cyc}\sin A$$

Proposed by Marian Ursărescu and Florică Anastase - Romania

JP.585. In $\Delta ABC, AA', BB', CC'$ are internal bisectors which intersect the circumcircle of triangle in the points A'', B'', C''. Prove that:

$$6\left(\left(\frac{R}{r}\right)^2-2\right)\le\frac{AA''}{A'A''}+\frac{BB''}{B'B''}+\frac{CC''}{C'C''}\le\frac{1}{3}\left(7-\frac{2r}{R}\right)^2$$

Proposed by Marian Ursărescu and Florică Anastase - Romania

PROBLEMS FOR SENIORS

SP.571. For given $n\ge 3$, prove that 3 is the largest positive value of the constant k such that:

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}-n\ge k(a_1+a_2+\ldots+a_n-n)$$

for any $a_1\ge a_2\ge\ldots\ge a_{n-1}\ge 1\ge a_n>0$ with

$$a_1a_2+a_2a_3+\ldots+a_{n-1}a_n+a_na_1=n.$$

Proposed by Vasile Cîrtoaje - Romania

SP.572. Let a, b, c be positive real numbers such that $a=\min\{a,b,c\}$ and $a^4bc\ge 1$, and let

$$F(a,b,c)=\sqrt[3]{abc}-\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Prove that:

$$F(a,b,c)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

Proposed by Vasile Cîrtoaje - Romania

SP.573. Let $\lambda \geq \frac{1}{2}$ fixed. If $a, b, c > 0, a + b + c \leq 4$ then find the maximum value of

$$P = \frac{\sqrt{abc}}{(a+b)(b+c)} - \frac{\lambda}{c(a+b)}$$

Proposed by Marin Chirciu - Romania

SP.574. If $a, b, c, d > 0; a^2 + b^2 + c^2 + d^2 = 4$ then:

$$\frac{1}{(1+ab)^3}+\frac{1}{(1+ac)^3}+\frac{1}{(1+ad)^3}+\frac{1}{(1+bc)^3}+\frac{1}{(1+bd)^3}+\frac{1}{(1+cd)^3} \geq \frac{3}{4}$$

Proposed by Daniel Sitaru - Romania

SP.575. If $x, y, z > 0$, then prove that:

$$\frac{\sum x^2 - \sum xy}{8(\sum x)^2} + \left(\sum x\right) \sum \frac{1}{2x+y+z} \leq \frac{3(\sum x)^2}{4\sum xy}$$

Proposed by Neculai Stanciu - Romania

SP.576. If $a, b, c \geq 1$, then:

$$\sqrt[3]{\frac{a^3+b^3+c^3}{3}} - \frac{a+b+c}{3} \geq \sqrt[3]{\frac{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}{3}} - \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}$$

Proposed by Vasile Mircea Popa - Romania

SP.577. Find all $n \in \mathbb{N}^*$ such that:

$$\int_0^1 (\sin x)^{2n-2} \cdot (\cos x)^{2n} dx \geq \frac{1}{4^{1011}}$$

Proposed by Daniel Sitaru - Romania

SP.578. If $x, y \in (0, \frac{\pi}{2})$ then:

$$\log^2_{\sin x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) + \log^2_{\cos x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) \geq 2$$

Proposed by Daniel Sitaru - Romania

SP.579. If $f : [a, b] \to \mathbb{R}; 0 < a \leq b; f$ - continuous then:

$$\frac{b^{4047} - a^{4047}}{4047} + \int_a^b f^2(x^{2024}) dx \geq \frac{1}{1012} \int_{a^{2024}}^{b^{2024}} f(x) dx$$

Proposed by Daniel Sitaru - Romania

SP.580. If $a \in \mathbb{R}; m, n \geq 1$ then:
$$(2+\cos a)^{m+n} + (3+\cos a)^m + (3+\cos a)^n \leq (3+\cos a)^{m+n} + 1$$
Proposed by Daniel Sitaru - Romania

SP.581. If $x, y, z \geq 1$ then:
$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+xyz\right)\left(\frac{1}{x}+\frac{2}{y}+xy^2\right) \geq \left(x+y+z+\frac{1}{xyz}\right)\left(x+2y+\frac{1}{xy^2}\right)$$
Proposed by Daniel Sitaru - Romania

SP.582. If $0 < a \leq b$ then:
$$\int_a^b (\sin x)^{2\sin^2 x} \cdot (1-\sin^2 x)^{1-\sin^2 x} dx \geq \frac{b-a}{2}$$
Proposed by Daniel Sitaru - Romania

SP.583. Prove that if $x, y, z \geq 1$ then:
$$2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) + xy + yz + zx \geq 2(x+y+z) + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}$$
Proposed by Daniel Sitaru - Romania

SP.584. Let be the sequence $(x_n)n \geq 1$ defined by $x_1 = 1$, $x_{n+2} = 3x_{n+1} - x_n, \forall n \in \mathbb{N}$. Find:
$$L = \lim_{n\to\infty} \sqrt[n]{\sum_{k=0}^{0} \frac{x_{2k+1}}{x_k + x_{k+1}}}$$
Proposed by Marian Ursărescu and Florică Anastase - Romania

SP.585. Let $f : [n-1, n] \to [n, n+1]$ be a continuous function such that:
$$\int_{n-1}^{n} (1 + xf'(x))dx \leq nf(n) - (n-1)f(n-1),$$
then prove:
$$\int_{n-1}^{n} \frac{dx}{f(x)} \leq \frac{2}{n+1}, n \in \mathbb{N}^*$$
Proposed by Marian Ursărescu and Florică Anastase - Romania

UNDERGRADUATE PROBLEMS

UP.571. Prove that is the least value of the constant $k > 2$ such that:

$$(a^k+b^k+c^k)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 9$$

for any positive real numbers a, b, c with at most one of them less than 1 and

$$a^5+b^5+c^5=3.$$

Proposed by Vasile Cîrtoaje - Romania

UP.572. Prove that 25/17 is the largest positive value of the constant k such that:

$$\frac{1}{a^2+k}+\frac{1}{b^2+k}+\frac{1}{c^2+k}+\frac{1}{d^2+k}\geq\frac{4}{1+k}$$

for any nonnegative real numbers a, b, c, d with at most one of them larger than 1 and

$$ab+ac+ad+bc+bd+cd=6.$$

Proposed by Vasile Cîrtoaje - Romania

UP.573. Let be $\lambda > 1$ fixed. Find:

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{k+1}{\lambda^k}$$

Proposed by Marin Chirciu - Romania

UP.574. If $0 \leq a \leq b \leq 1$ then:

$$\int_a^b\int_a^b\int_a^b\frac{dxdydz}{\sqrt{1+xyz}}\geq(b-a)^2\int_a^b\frac{dx}{\sqrt{1+x^3}}$$

Proposed by Daniel Sitaru - Romania

UP.575. Calculate:

$$2\int_0^1\int_0^\infty\frac{y(y+1)}{(x^2+(y+1)^2)^2}dydx$$

Proposed by Said Attaoui - Algeria

UP.576. Find a closed form:

$$\int_0^\infty\frac{\arctan(x)}{\sqrt[5]{x^{10}+1}}dx$$

Proposed by Vasile Mircea Popa - Romania

UP.577. Find a closed form:

$$\int_0^1 \frac{x^2 \ln x}{x^3 + x\sqrt{x} + 1} dx$$

Proposed by Vasile Mircea Popa - Romania

UP.578. Prove the equality:

$$\int_0^\infty \frac{x^4\sqrt{x}\ln x}{(x+1)(x^2+1)(x^4+1)} dx = \frac{1}{8}\pi^2\sqrt{5 - \frac{7}{\sqrt{2}}}$$

Proposed by Vasile Mircea Popa - Romania

UP.579. Find a closed form:

$$\int_{-1}^1 \frac{1}{\sqrt[5]{(1-x)^2(1+x)^3}} dx$$

Proposed by Vasile Mircea Popa - Romania

UP.580. If $f : [a,b] \to [-1,\infty); a,b \in \mathbb{R}; a \le b; f$ continuous, then:

$$\left(\int_a^b (1+f(x))dx\right)^3 \ge (b-a)^3 + 3(b-a)^2 \int_a^b f(x)dx$$

Proposed by Daniel Sitaru - Romania

UP.581. If $1 \le a \le b$ then:

$$24(b-a)^2 \ln\frac{b}{a} + (b^2-a^2)^3 \ge 12(b-a)^2(b^2-a^2) + 24\ln^3\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru - Romania

UP.582. Prove that exists $X \in M_{2,3}(\mathbb{R}); Y \in M_{3,2}(\mathbb{R})$ such that:

$$X \cdot Y = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}; Y \cdot X = \begin{pmatrix} 2 & 6 & 6 \\ 3 & 9 & 9 \\ -3 & -9 & -9 \end{pmatrix}$$

Proposed by Daniel Sitaru - Romania

UP.583. Solve for complex numbers:

$$x^8 + 4x^7 + 22x^6 + 52x^5 + 69x^4 + 56x^3 + 28x^2 + 8x + 1 = 0$$

Proposed by Daniel Sitaru - Romania

UP.584. Let $a,b,c,d>1$ and $f:[a,b]\to[c,d]$ a continuous function for which $\exists\lambda\in(a,b)$ such that:

$$a\int_a^\lambda f(x)dx+b\int_\lambda^b f(x)dx\ge a+c$$

then prove:

$$\int_a^b\frac{x}{f(x)}dx\le\Big(\frac{1}{a}+\frac{1}{b}\Big)\frac{b^2-a^2-2}{2}$$

Proposed by Marian Ursărescu and Florică Anastase - Romania

UP.585. If $f:(0,\infty)\to(0,\infty)$; f continuous and $f(\frac{1}{x})+f(\frac{1}{y})=2f(\frac{1}{x+y})$; $(\forall)x,y>0$ then $(\forall)a,b>0$:

$$\int_a^b\int_a^b\int_a^b f\Big(\frac{1}{x+y+z}\Big)dxdydz=(b-a)^2\int_a^b f\Big(\frac{1}{x}\Big)dx$$

Proposed by Daniel Sitaru - Romania

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