ROMANIAN MATHEMATICAL MAGAZINE

Solve for positive numbers:

$$\begin{cases} x\sqrt[3]{x}+y\sqrt[3]{y}=162 \\ x\sqrt[3]{y}+y\sqrt[3]{x}=162 \end{cases}$$

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$$Let\ a^3=x, b^3=y\ then\ above\ equation\ can\ be\ written\ as$$

$$a^4+b^4=162\ (1)$$
$$ab(a^2+b^2)=162\ (2)$$

$$a^4+b^4=162\ or\ (a^2+b^2)^2-2a^2b^2=162\ or\ s^2-2p^2\overset{s=a^2+b^2,p=ab}{=}162$$

$$From\ (2)\ ps=162\ or\ s=\frac{162}{p}$$

$$s^2-2p^2=162\ or\ \left(\frac{162}{p}\right)^2-2p^2=162\ or\ 2p^4+162p^2-162^2=0$$

$$p^4+81p^2-13122=0\ or\ t^2+81t-13122\overset{t=p^2}{=}0$$

$$t=p^2=\frac{-81\pm 243}{2}\ or\ t=p^2=81\ (taking\ \text{positive}\ value)\ or$$

$$p=9\ and\ s=\frac{162}{p}=18$$

$$p=a^2+b^2=18, s=ab=9$$

$$(a-b)^2=a^2+b^2-2ab=18-18=0\ or\ a=b$$

$$then\ a=b=\sqrt{9}=3\ and\ x=a^3=27, y=b^3=27$$