ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda > 1$ then solve for real numbers:

$$(\lambda^{2x}+\lambda^{2y}+\lambda^{2z})\left(\frac{1}{\lambda^x}+\frac{1}{\lambda^y}+\frac{1}{\lambda^z}\right)=3$$

$$3(\lambda^{x+y}+\lambda^{y+z}+\lambda^{z+x})=\lambda^x+\lambda^y+\lambda^z$$

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$$Let\ \lambda^x=a,\lambda^y=b,\lambda^z=c\ clearly\ a,b,c>0\ as\ \lambda>1$$
$$then\ given\ equation\ can\ be\ written\ as:$$
$$(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3\ \ (1)$$

$$3(ab+bc+ca)=a+b+c\ (2)$$

$$Let\ \ S=a+b+c,P=ab+bc+ca,R=abc\ \ from\ (2)we\ get\ 3P=S$$

$$From\ (1)we\ get\ \ (a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3$$

$$\left((a+b+c)^2-2(ab+bc+ca)\right)\left(\frac{ab+bc+ca}{abc}\right)=3$$

$$(S^2-2P)\frac{P}{R}=3\ or,R=\frac{P}{3}(S^2-2P)\overset{S=3P}{=}\frac{P(9P^2-2P)}{3}\ \ (3)$$

$$Using\ A.M-G.M\ inequality\ we\ get:$$

$$\left(\frac{a+b+c}{3}\right)^3\geq abc\ or\ S^3\geq 27R\ or,R\overset{S=3P}{\leq}P^3$$

$$\frac{P(9P^2-2P)}{3}\overset{(3)}{\leq}P^3\ or\ \ 6P^2\overset{P=abc>0}{\leq}2P\ or\ P\leq\frac{1}{3}\ \ so\ S=3P\leq 3\times\frac{1}{3}\leq 1\ (4)$$

$$Well\ known\ inequality\ (a+b+c)^2\geq 3(ab+bc+ca)$$

$$or,S^2\geq 3P or,S^2\overset{S=3P}{\geq}3.\frac{S}{3}\ or,S\overset{S=a+b+c>0}{\geq}1\ (5)\ now\ from(4)\ \&\ (5)$$

$$\boldsymbol{We\ get\ S \geq 1\ \&\ S \leq 1},$$

$$\boldsymbol{so\ S = 1\ or\ a + b + c = 1\ and\ ab + bc + ca = P = \frac{S}{3} = \frac{1}{3}}$$

$$\boldsymbol{(a-b)^2 + (b-c)^2 + (c-a)^2 = 2(a^2 + b^2 + c^2) - 2(ab + bc + ca)}$$

$$\boldsymbol{= 2(S^2 - 2P) - 2P = 2(s^2 - 3P) \overset{S=1, P=\frac{1}{3}}{=} 2(1-1) = 0}$$

$$\boldsymbol{a - b = 0 \Rightarrow a = b, (b-c) = 0 \Rightarrow b = c, (c-a) = 0 \Rightarrow c = a}$$

$$\boldsymbol{a = b = c = \frac{1}{3}(as\ S = a + b + c = 1)}$$

$$\boldsymbol{\lambda^x = a = \frac{1}{3}\ or, x = \log_\lambda\left(\frac{1}{3}\right), \lambda^y = b = \frac{1}{3}\ or\ y = \log_\lambda\left(\frac{1}{3}\right)},$$

$$\boldsymbol{\lambda^z = c = \frac{1}{3}\ or\ z = \log_\lambda\left(\frac{1}{3}\right)}$$

$$\boldsymbol{Solution: x = y = z = \log_\lambda\left(\frac{1}{3}\right)}$$