ROMANIAN MATHEMATICAL MAGAZINE

Solve for reals:

$$\begin{cases}\sqrt{x+\frac{1}{5}}+\sqrt{y+\frac{1}{5}}=\sqrt{15}\\ x+y=3+8\sqrt{xy}\end{cases}$$

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$$Let\ a=\sqrt{x+\frac{1}{5}},b=\sqrt{x+\frac{1}{5}}\ then\ x=a^2-\frac{1}{5},y=b^2-\frac{1}{5}$$

$$According\ to\ the\ given\ condition, a+b=\sqrt{15}\ (1)$$

$$x+y=(a^2+b^2)-\frac{2}{5}=(a+b)^2-2ab-\frac{2}{5}\overset{(1)}{=}15-2ab-\frac{2}{5}=\frac{73}{5}-2ab\ (2)$$

$$again,\quad x+y=3+8\sqrt{xy}\ (3) eliminate\ (x+y)\ from\ (2)\ \&\ (3) we\ get,$$

$$3+8\sqrt{xy}=\frac{73}{5}-2ab\ or\ \sqrt{xy}=\frac{\frac{29}{5}-ab}{4}$$

$$or\ xy=\frac{1}{16}\left(\frac{29}{5}-ab\right)^2=\frac{1}{16}\left(\frac{841}{25}-\frac{58}{5}ab+a^2b^2\right)\ (4)$$

$$xy=\left(a^2-\frac{1}{5}\right)\left(b^2-\frac{1}{5}\right)=a^2b^2-\frac{a^2+b^2}{5}+\frac{1}{25}=a^2b^2-\frac{(a+b)^2-2ab}{5}+\frac{1}{25}\overset{(1)}{=}$$

$$=a^2b^2+\frac{2}{5}ab-\frac{74}{25}\ (5)\ from\ (4) and\ (5) eliminate\ xy\ we\ get:$$

$$\frac{1}{16}\left(\frac{841}{25}-\frac{58}{5}ab+a^2b^2\right)=a^2b^2+\frac{2}{5}ab-\frac{74}{25}$$

$$or\left(\frac{841}{25}-\frac{58}{5}ab+a^2b^2\right)=16a^2b^2+\frac{32}{5}ab-\frac{74}{25}or$$

$$841-290ab+25a^2b^2=400a^2b^2+16ab-1184$$

$$or\ 5a^2b^2+6ab-27=0\ or\ ab=\frac{-6\pm\sqrt{36+540}}{10}=\frac{9}{5},-3$$

$$as\ ab\geq 0\ so\ ab\neq -3, ab=\frac{9}{5}$$

$$Now\ xy\overset{(4)}{=}\frac{1}{16}\left(\frac{29}{5}-ab\right)^2\overset{ab=\frac{9}{5}}{=}1\ and\ x+y\overset{(3)}{=}3+8\sqrt{xy}\overset{xy=1}{=}11$$

$$We\ construct\ an\ equation\ with\ roots\ x,y:$$

$$m^2-(x+y)m+xy=0\ or, m^2-11m+1=0\ or, m=\frac{11\pm\sqrt{117}}{2}=\frac{11\pm 3\sqrt{13}}{2}$$

$$solution\ (x,y)=\left(\frac{11+3\sqrt{13}}{2},\frac{11-3\sqrt{13}}{2}\right)\ or\left(\frac{11-3\sqrt{13}}{2},\frac{11+3\sqrt{13}}{2}\right)$$