ROMANIAN MATHEMATICAL MAGAZINE

Let be $\lambda \geq 1$ fixed. Solve for reals:

$$\begin{cases} (x+y)(x^2+y^2) = \lambda(4\lambda^2+1) \\ (x-y)(x^2-y^2) = 2\lambda \end{cases}$$

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$$Let\ u = x+y, v = x-y\ then\ x = \frac{u+v}{2}, y = \frac{u-v}{2}, xy = \frac{u^2-v^2}{4}$$

$$(x+y)(x^2+y^2) = \lambda(4\lambda^2+1)\ or\ \ (x+y)((x+y)^2-2xy) = \lambda(4\lambda^2+1)$$

$$u\left(u^2-\frac{u^2-v^2}{2}\right) = \lambda(4\lambda^2+1)\ or\ , u(u^2+v^2) = 2\lambda(4\lambda^2+1)\ (1)$$

$$(x-y)(x^2-y^2) = 2\lambda\ \ or\ (x-y)^2(x+y) = 2\lambda\ or\ uv^2 = 2\lambda\ or\ v^2 = \frac{2\lambda}{u}\ (2)$$

$$Eliminating\ v^2 between\ (1)\&\ (2) we\ get:$$

$$u\left(u^2+\frac{2\lambda}{u}\right) = 2\lambda(4\lambda^2+1)\ or, u^3+2\lambda = 8\lambda^3+2\lambda\ or, u^3 = 8\lambda^3\ or$$

$$u = 2\lambda\ (as\ \lambda \geq 1)\ \&\ v^2 = \frac{2\lambda}{u} = \frac{2\lambda}{2\lambda} = 1\ or\ \ v = \pm 1$$

$$x = \frac{u+v}{2} = \frac{2\lambda \pm 1}{2} = \lambda \pm \frac{1}{2}, y = \frac{u-v}{2} = \frac{2\lambda \mp 1}{2} = \lambda \mp \frac{1}{2}$$

$$Solutions: (x,y) = \left(\lambda+\frac{1}{2}, \lambda-\frac{1}{2}\right), \left(\lambda-\frac{1}{2}, \lambda+\frac{1}{2}\right)$$