ROMANIAN MATHEMATICAL MAGAZINE

Let be $\lambda \in R^+$ fixed. Solve for reals:

$$\begin{cases} x^6 - y^6 = \lambda^6 - 1 \\ x^4 + x^2y^2 + y^4 = \lambda^4 + \lambda^2 + 1 \end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ a = x^2, b = y^2 (a, b \geq 0)$$
$$x^6 - y^6 = \lambda^6 - 1\ or, (x^2 - y^2)(x^4 + x^2y^2 + y^4) = (\lambda^2)^3 - (1)^3$$

$$(a - b)(a^2 + ab + b^2) = (\lambda^2 - 1)(\lambda^4 + \lambda^2 + 1)$$

$$Comparing\ we\ get:$$
$$a - b = \lambda^2 - 1\ (1),\ a^2 + ab + b^2 = \lambda^4 + \lambda^2 + 1\ \ (2)$$

$$Let\ a + b = s, d = a - b = \lambda^2 - 1\ \ then\ from\ (2)\ we\ get:$$
$$a^2 + ab + b^2 = \lambda^4 + \lambda^2 + 1$$

$$(a + b)^2 - 2ab + ab = \lambda^4 + \lambda^2 + 1\ \ or\ (a + b)^2 - ab = \lambda^4 + \lambda^2 + 1$$
$$(a + b)^2 - \frac{1}{4}((a + b)^2 - (a - b)^2) = \lambda^4 + \lambda^2 + 1$$
$$s^2 - \frac{1}{4}(s^2 - d^2) = \lambda^4 + \lambda^2 + 1$$
$$\frac{3s^2}{4} = \lambda^4 + \lambda^2 + 1 - \frac{d^2}{4}\ or\ 3s^2 \overset{d=a-b=\lambda^2-1}{=} 4(\lambda^4 + \lambda^2 + 1) - (\lambda^2 - 1)^2$$

$$s^2 = (\lambda^2 + 1)^2\ or\ s = a + b = \lambda^2 + 1\ \ (as\ a, b \geq 0)\ \ (3)$$

$$ab = \frac{1}{4}((a + b)^2 - (a - b)^2) = \frac{1}{4}((\lambda^2 + 1)^2 - (\lambda^2 - 1)^2) = 4\frac{\lambda^2}{4} = \lambda^2\ (4)$$

$$We\ construct\ an\ equation\ with\ roots\ a\ \&\ b$$
$$t^2 - (a + b)t + ab = 0\ or\ t^2 - (\lambda^2 + 1)t + \lambda^2 = 0\ or\ t = \frac{(\lambda^2 + 1) \pm (\lambda^2 - 1)}{2}$$
$$or\ t = \lambda^2, 1 \Rightarrow\ x^2 = a = \lambda^2\ or\ x = \pm\lambda\ \&\ y^2 = b = 1\ or\ y = \pm 1$$

$$Solutions: (x, y) \in \{(\lambda, 1), (-\lambda, 1), (-\lambda, -1), (\lambda, -1)\}$$