ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},\qquad ab+bc+ca=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

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$$Let\ a+b+c=p, ab+bc+ca=q, abc=r$$

$$then\ \ a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ or, a+b+c=\frac{ab+bc+ca}{abc}\ \ or, p=\frac{q}{r}\ \ or, pr=q\ (1)$$

$$ab+bc+ca=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\ \ \ or, ab+bc+ca=\frac{a+b+c}{abc}\ \ or,$$

$$q=\frac{p}{r}\ or, qr=p\ (2)$$

$$From\ (2), qr=p\ or, (pr).r\overset{(1)}{=}p\ \ or, p(r^2-1)=0\ which\ implies\ r=1,-1$$

$$p=0\ not\ possible\ because$$

$$p=\ \ a+b+c=0\ gives\ \ from\ (1)\ q=ab+bc+ca=0$$

$$as\ a\ result\ (a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\ \ or, a^2+b^2+c^2=0$$

$$\Rightarrow\ \ a=b=c=0, for\ this\ value\ reciprocal\ \ are\ not\ defined\ , so\ p\neq 0$$

$$Case-1{:}\ \ r=1\ then\ from(1) we\ get\ p=q$$

$$known\ result\ (a-1)(b-1)(c-1)=abc-(ab+bc+ca)+(a+b+c)-1$$

$$=r-q+p-1=1-p+p-1=0$$

$$or, (a-1)(b-1)(c-1)$$

$$=0\ which\ implies\ that\ at\ least\ one\ of\ \ the\ number\ a, b, c$$

$$is\ equal\ to\ 1\ . If\ \ a=1\ then\ r=abc=1\Rightarrow bc=1\ or, c=\frac{1}{b}$$

$$solution\ for\ Case-1, all\ the\ permutation\ \left(1, t, \frac{1}{t}\right), t\in\ \ R-\{0\}$$

$$Case-2{:}\ r=-1\ then\ from(1) we\ get\ q=-p$$

$$Known\ result\ (a+1)(b+1)(c+1)=abc+(ab+bc+ca)+(a+b+c)+1$$

$$=r+q+p+1=-1-p+p+1=0\ or, (a+1)(b+1)(c+1)=0$$

$$which\ implies\ that\ at\ least\ one\ of\ \ the\ number\ a, b, c\ is\ equal\ to-1$$

$$If\ \ a=-1\ then\ r=abc\Rightarrow -1=-bc\ or, c=\frac{1}{b}$$

$$Ssolution\ for\ Case-2, all\ the\ permutation\ \left(-1, t, \frac{1}{t}\right), t\in\ \ R-\{0\}$$