ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $x^2 + y^2 = 16$, $y^2 + z^2 = 48$, $y^2 - zx = 0$

then find $xy + yz + zx$.

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Geometric Approach:

Consider a right-angled triangle ABC with $\angle A = 90°$, $\angle B = 60°$, and $\angle C = 30°$. Let $AP \perp BC$ be the altitude drawn to the hypotenuse. Let $AP = y$, $BP = x$, and $PC = z$.

By the Pythagorean theorem in $\triangle ABP$ and $\triangle ACP$:

$$AB = \sqrt{x^2 + y^2} = \sqrt{16} = 4,\quad AC = \sqrt{y^2 + z^2} = \sqrt{48} = 4\sqrt{3}.$$

From the geometric properties of the altitude to the hypotenuse:

$y^2 = xz \Rightarrow y^2 - xz = 0$. The area of $\triangle ABC$ can be calculated as:

$$\Delta = \tfrac{1}{2} \cdot AB \cdot AC = \tfrac{1}{2} \cdot 4 \cdot 4\sqrt{3} = 8\sqrt{3}.$$

Also, splitting the area into $\Delta_1 = S(\triangle ABP)$ and $\Delta_2 = S(\triangle ACP)$: $\Delta_1 = \tfrac{1}{2}xy$, $\Delta_2 = \tfrac{1}{2}yz$.

$\Delta_1 + \Delta_2 = \tfrac{1}{2}(xy + yz) = 8\sqrt{3} \Rightarrow xy + yz = 16\sqrt{3}$. In $\triangle ABC$, since $AB = 4$ and $\angle B = 60°$:

$y = AB \cdot \sin 60° = 4 \cdot (\sqrt{3}/2) = 2\sqrt{3}$. Thus, $zx = y^2 = (2\sqrt{3})^2 = 12$.

Combining the results: $xy + yz + zx = 16\sqrt{3} + 12$.

Algebraic Approach:

From the given system: (1) $x^2 + y^2 = 16$, (2) $y^2 + z^2 = 48$, (3) $y^2 = xz$

Subtracting (1) from (2): $z^2 - x^2 = 32 \Rightarrow (z - x)(z + x) = 32$.

From (1) and (3), substitute $y^2 = xz$ into (1): $x^2 + xz = 16 \Rightarrow x(x + z) = 16 \Rightarrow x + z = 16/x$.

Substitute $x + z = 16/x$ into $z^2 - x^2 = 32$: $(z - x) \cdot (16/x) = 32 \Rightarrow z - x = 2x \Rightarrow z = 3x$.

Substituting $z = 3x$ into (3): $y^2 = x(3x) = 3x^2$.

Substitute $y^2 = 3x^2$ into (1): $x^2 + 3x^2 = 16 \Rightarrow 4x^2 = 16 \Rightarrow x^2 = 4 \Rightarrow x = 2$ $(x > 0)$.

Finding y and z: $z = 3(2) = 6$, $y = \sqrt{3x^2} = \sqrt{3 \cdot 4} = 2\sqrt{3}$.

Calculating $xy + yz + zx$: $xy = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$, $yz = 2\sqrt{3} \cdot 6 = 12\sqrt{3}$, $zx = 6 \cdot 2 = 12$

$$xy + yz + zx = 4\sqrt{3} + 12\sqrt{3} + 12 = 12 + 16\sqrt{3}.$$