ROMANIAN MATHEMATICAL MAGAZINE

Let n be a positive real fixed number. Solve for reals:

$$\begin{cases} a+b+c=n \\ ab+bc+ca=\dfrac{n^2-n+1}{3}+a \end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Adgozel Adgozalov

$$\begin{cases} a+b+c=n \\ ab+bc+ca=\dfrac{n^2-n+1}{3}+a \end{cases} \implies \begin{cases} b+c=n-a \\ a(b+c)+bc=\dfrac{n^2-n+1}{3}+a \end{cases}$$

$$\implies a(n-a)+bc=\frac{n^2-n+1}{3}+a$$

$$\implies bc=a^2-an+a+\frac{n^2-n+1}{3}$$

Let $t_1=b$ and $t_2=c$. Then b,c are roots of the quadratic equation:

$$t^2-(n-a)t+\left(a^2-an+a+\frac{n^2-n+1}{3}\right)=0$$

Since b,c ⍰R, we must have $\Delta\geq 0$:

$$\begin{aligned}\Delta &= (n-a)^2-4\left(a^2-an+a+\frac{n^2-n+1}{3}\right)\\ &= n^2-2an+a^2-4a^2+4an-4a-\frac{4n^2-4n+4}{3}\\ &= -3a^2+2an-4a-\frac{n^2-4n+4}{3}\\ &= -\frac{1}{3}\left[9a^2-6a(n-2)+(n-2)^2\right]\\ &= -\frac{1}{3}\left(3a-(n-2)\right)^2\geq 0\end{aligned}$$

This implies:

$$3a-(n-2)=0 \implies 3a=n-2 \implies a=\frac{n-2}{3}$$

Since $\Delta=0$, we have $b=c$:

$$2b=n-\frac{n-2}{3}=\frac{2n+2}{3} \implies b=c=\frac{n+1}{3}$$

Thus, the real solution is:

$$\therefore (a,b,c)=\left(\frac{n-2}{3},\frac{n+1}{3},\frac{n+1}{3}\right)$$