ROMANIAN MATHEMATICAL MAGAZINE

$$\begin{cases} a = \dfrac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \\ b = \dfrac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \end{cases}$$

Express x and y in terms of a and b

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$$\begin{cases} a = \dfrac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \\ b = \dfrac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \end{cases} \Rightarrow \begin{cases} a = \dfrac{x}{\sqrt{1 - x^2 + y^2}} - \dfrac{y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \\ b = \dfrac{y}{\sqrt{1 - x^2 + y^2}} - \dfrac{x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \end{cases}$$

$$k = \frac{1}{\sqrt{1 - x^2 + y^2}}, \qquad m = \frac{\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}}$$

$$\begin{cases} kx - my = a \\ -mx + ky = b \end{cases} \Rightarrow \begin{pmatrix} k & -m \\ -m & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow A \cdot B = C$$

$$A = \begin{pmatrix} k & -m \\ -m & k \end{pmatrix} \to Det(A) = k^2 - m^2 = \frac{1}{1 - x^2 + y^2} - \frac{x^2 - y^2}{1 - x^2 + y^2} = 1 \neq 0$$

$$A^{-1} \cdot C = B \Rightarrow A^{-1} = \frac{1}{Det(A)}\begin{pmatrix} k & m \\ m & k \end{pmatrix} \Rightarrow \begin{pmatrix} k & m \\ m & k \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} ka + mb = x \\ ma + kb = y \end{cases} \Rightarrow x^2 - y^2 = (ka + mb)^2 - (ma + kb)^2$$
$$= a^2(k^2 - m^2) - b^2(k^2 - m^2)$$

$$x^2 - y^2 = Det(A) \cdot (a^2 - b^2) = a^2 - b^2$$

$$k = \frac{1}{\sqrt{1 - x^2 + y^2}} = \frac{1}{\sqrt{1 - a^2 + b^2}}, \quad m = \frac{\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} = \frac{\sqrt{a^2 - b^2}}{\sqrt{1 - a^2 + b^2}}$$

$$\textit{Therefore} \quad \begin{cases} ka + mb = x \\ ma + kb = y \end{cases} \Rightarrow \begin{cases} x = \dfrac{a + b\sqrt{a^2 - b^2}}{\sqrt{1 - a^2 + b^2}} \\ y = \dfrac{b + a\sqrt{a^2 - b^2}}{\sqrt{1 - a^2 + b^2}} \end{cases}$$