ROMANIAN MATHEMATICAL MAGAZINE

If $\begin{cases} x^2+y^2+2x+4y-20=0 \\ z^2+w^2-2w-143=0 \\ xw+yz-x+w+2z-61\geq 0 \end{cases}$ $S=\frac{y^2}{25}+\frac{w^2}{144}$ then find:

$$S_{\min}+S_{\max}=?$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

The first two equations define the a couple of circles.

$$Equation\ 1.\quad x^2+y^2+2x+4y-20=0\to\ (x+1)^2+(y+2)^2=5^2$$

$$M_1(-1;-2)\quad R_1=5$$

$$Equation\ 2.\quad z^2+w^2-2w-143=0\to z^2+(w-1)^2=12^2$$

$$M_2(0;1)\quad R_2=12$$

$$(x-a)^2+(y-b)^2=R^2\ \to\ \begin{cases} x=a+Rcos(\alpha) \\ y=b+Rsin(\alpha) \end{cases}$$

$$\begin{cases} x+1=5\cos(\alpha) \\ y+2=5\sin(\alpha) \end{cases}\to\ \begin{cases} x=5\cos(\alpha)-1 \\ y=5\sin(\alpha)-2 \end{cases}$$

$$\begin{cases} z=12\cos(\beta) \\ w-1=12\sin(\beta) \end{cases}\ \to\ \ w=12\sin(\beta)+1$$

$$xw+yz-x+w+2z-61\geq 0\to\ x(w-1)+z(y+2)+w-1\geq 60$$

$$(5\cos(\alpha)-1)12\sin(\beta)+12\cos(\beta).5\sin(\alpha)+12\sin(\beta)\geq 60$$

$$60\sin(\alpha)\cos(\beta)+60\sin(\beta)\cos(\alpha)\geq 60$$

$$\sin(\alpha+\beta)\geq 1\to(\alpha+\beta)\in\left[\frac{\pi}{2}+\pi k\right]\ \ \sin(\beta)=\cos(\alpha)$$

$$S=\frac{y^2}{25}+\frac{w^2}{144}=\left(\frac{y}{5}\right)^2+\left(\frac{w}{12}\right)^2=\left(\frac{5\sin(\alpha)-2}{5}\right)^2+\left(\frac{12\cos(\alpha)-1}{12}\right)^2=$$

$$=\sin^2(\alpha)-\frac{4}{5}\sin(\alpha)+\frac{4}{25}+\cos^2(\alpha)-\frac{1}{6}\cos(\alpha)+\frac{1}{144}$$

$$=\frac{4201}{3600}+\frac{1}{6}\cos(\alpha)-\frac{4}{5}\sin(\alpha)$$

$$S = C + \frac{1}{6}\cos(\alpha) - \frac{4}{5}\sin(\alpha) \quad Amplitude \to \quad A = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{\sqrt{601}}{30}$$

$$S_{\max} = C + A \quad S_{\min} = C - A \quad S_{\max} + S_{\min} = 2C = 2.\frac{4201}{3600} = \frac{4201}{1800}$$