ROMANIAN MATHEMATICAL MAGAZINE

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f^2(x)+2yf(x)+f(y)=f\big(y+f(x)\big)\ \forall\ x,y\in\mathbb{R}$$

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$$f^2(x)+2yf(x)+f(y)=f\big(y+f(x)\big)\ \forall\ x,y\in R\ (1)$$

$Suppose\ f(x)=c\ is\ constant\ then\ the\ equation\ becomes$

$c^2+2yc+c=c\ \forall\ y\ ,this\ can\ only\ be\ true\ if\ c=0$

$Hence\ one\ solution\ is\ f(x)\equiv 0$

$From\ (1) we\ have$

$$f\big(y+f(x)\big)-f(y)=f^2(x)+2yf(x)+f(y)$$

$R.H.S\ part\ has\ the\ same\ form\ as\ the\ algebraic\ identity$

$$(y+a)^2-y^2=2ay+a^2$$

$this\ suggest\ that\ f(y)\ behave\ like\ y^2+some\ constant.$

$Let\ g(y)=\ f(y)-y^2\ or\ f(y)=g(y)+y^2\ (2)$

$where\ g(y)\ represent\ the\ non\ quadratic\ part\ of\ f(y).$

$From\ (1) we\ have\ f\big(y+f(x)\big)=f(y)+f^2(x)+2yf(x)+f(y)$

$Using\ (2) we\ \ have:$

$$g\big(y+f(x)\big)+\big(y+f(x)\big)^2=g(y)+y^2+(g(x)+x^2)^2+2y(g(x)+x^2)$$

$$g\big(y+f(x)\big)+y^2+f^2(x)+2yf(x)=g(y)+y^2+(g(x)+x^2)^2+2y(g(x)+x^2)$$

$$g\big(y+f(x)\big)+(g(x)+x^2)^2+2y(g(x)+x^2)+2yf(x)\overset{f(x)=g(x)+x^2\ from\ (2)}{=}$$

$$=g(y)+(g(x)+x^2)^2+2y(g(x)+x^2)$$

$$g\big(y+f(x)\big)=g(y)$$

$This\ equation\ says\ that\ g(y)\ does\ not\ change\ when\ its\ argument\ increase\ by\ f(x)$

$since\ f\ is\ non\ constant\ it\ can\ take\ infinitely\ many\ values\ so\ g(y)\ must$

$be\ constant\ g(y)=c\ hence\ f(y)=g(y)+y^2=c+y^2$

$So\ solutions\ are\ f(x)\equiv 0\ or\ f(x)=\ x^2+c.$