ROMANIAN MATHEMATICAL MAGAZINE

Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(f(x)+y^2+4)+4y=x+f^2(y+2)\ \forall\, x,y\in\mathbb{R}$$

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$$f(f(x)+y^2+4)+4y=x+f^2(y+2)\ \forall\, x,y\in R\ (1)$$

$$We\ put\ y=-2$$
$$then\ f(f(x)+8)-8=x+f^2(0)\quad (2)$$

$$From\ (2): f(u)=f(v)\Rightarrow\ f(f(u)+8)-8=f(f(v)+8)-8$$
$$or\ \ u+f^2(0)\overset{(2)}{=}v+f^2(0)\ or, u=v,\qquad so\ f\ is\ injective$$

$$f(f(x)+8)=x+8+f^2(0)\ ,so\ every\ real\ number\ is\ in\ the\ image\ of\ f\ ,$$
$$so\ f\ is\ surjective$$

$$Again\ f(f(x)+8)\ is\ a\ linear\ function\ of\ x\ ,since\ from(2)\ value\ of$$
$$f(f(x)+8)=x+8+f^2(0)\ .Clearly\ f\ is\ \ linear.$$

$$Let\ f(x)=ax+b\ then\ \ from\ (1)\ we\ have:$$
$$a(f(x)+y^2+4)+b+4y=x+(a(y+2)+b)^2$$
$$or\ a(ax+b+y^2+4)+b+4y=x+a^2y^2+4a^2+b^2+4a^2y+4ab+2aby$$
$$a^2x+ab+ay^2+4a+b+4y=x+a^2y^2+4a^2+b^2+4a^2y+4ab+2aby\ \ (3)$$

$$Equalizing\ coefficients\ of\ x\ \&\ y^2\ \ we\ get$$
$$a^2=1\ or, a=\pm1\ \&\ a^2=a\ or, a=0,1\ so\ combine\ results\ is\ a=1$$

$$Putting\ a=1\ in\ (3\ )\ we\ get$$
$$x+b+y^2+4+b+4y=x+y^2+4+b^2+4y+4b+2by$$
$$or\ 2b=2by+4b+b^2$$

$$Equalizing\ cofficients\ of\ y\ we\ have\ 2b=0\ or\ b=0$$
$$so\ f(x)=ax+b=x$$