

ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt{2 + \frac{3}{x^2}} + \sqrt{3 + \frac{2}{x^2}} = \sqrt{x^2 + \frac{2}{3}}$$

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$$\sqrt{2 + \frac{3}{x^2}} + \sqrt{3 + \frac{2}{x^2}} = \sqrt{x^2 + \frac{2}{3}} \quad \{x^2 = k\}, \quad \sqrt{2 + \frac{3}{k}} + \sqrt{3 + \frac{2}{k}} = \sqrt{k + \frac{2}{3}}$$

$$\sqrt{\frac{2k+3}{k}} + \sqrt{\frac{3k+2}{k}} = \sqrt{\frac{3k+2}{3}}, \quad \therefore \text{Let's multiply all of them by: } \sqrt{\frac{k}{3k+2}}$$

$$\sqrt{\frac{2k+3}{3k+2}} + 1 = \sqrt{\frac{k}{3}}, \quad \therefore \text{Let's replace it with "m" } m = \sqrt{\frac{k}{3}}, k = 3m^2$$

$$\sqrt{\frac{6m^2+3}{9m^2+2}} + 1 = m, \quad \frac{6m^2+3}{9m^2+2} = (m-1)^2$$

$$6m^2 + 3 = (m^2 - 2m + 1)(9m^2 + 2), \quad 6m^2 + 3 = 9m^4 - 18m^3 + 11m^2 - 4m + 2$$

$$9m^4 - 18m^3 + 5m^2 - 4m - 1 = 0, \quad (3m^2 - m + 1)(3m^2 - 5m - 1) = 0$$

$$\begin{cases} 3m^2 - m + 1 = 0, & (\Delta < 0) \\ 3m^2 - 5m - 1 = 0, & m = \frac{5 \pm \sqrt{37}}{6} \end{cases}$$

$$x^2 = 3 \left(\frac{5 + \sqrt{37}}{6} \right)^2 = \frac{62 + 10\sqrt{37}}{12}, \quad x = \pm \sqrt{\frac{31 + 5\sqrt{37}}{6}}$$