ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$x^4 - x^2 - 16x + 2 = 2\sqrt{x^2 + 4x}$$

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$$x^4 - x^2 - 16x + 2 = 2\sqrt{x^2 + 4x}$$
$$Domain\ need\ x^2 + 4x \geq 0\ so\ x \in (-\infty, -4] \cup [0, \infty)$$
$$(x^4 - x^2 - 16x + 2)^2 = \left(2\sqrt{x^2 + 4x}\right)^2$$
$$x^8 + (x^2 + 16x - 2)^2 - 2x^4(x^2 + 16x - 2) = 4x^2 + 16x$$
$$x^8 - 2x^6 - 32x^5 + 5x^4 + 32x^3 + 248x^2 - 80x + 4 = 0\ (1)$$
$$Let\ x^8 - 2x^6 - 32x^5 + 5x^4 + 32x^3 + 248x^2 - 80x + 4 =$$
$$= (x^4 - 16x + p)(x^4 - 2x^2 - 16x + q)$$
$$= x^8 - 2x^6 - 3x^5 + x^4(p + q) + x^2(256 - 2p) - x(16q + 16p) + pq$$

$$Comparing\ coefficient\ \ we\ get\ coefficient\ of\ x^2 \to 256 - 2p = 248\ or, p = 4$$
$$Constant\ term \to\ pq = 4\ or\ q = 1\ as\ p = 4$$
$$x^8 - 2x^6 - 32x^5 + 5x^4 + 32x^3 + 248x^2 - 80x + 4 =$$
$$= (x^4 - 16x + 4)(x^4 - 2x^2 - 16x + 1)$$

$$From\ (1)\ (x^4 - 16x + 4)(x^4 - 2x^2 - 16x + 1) = 0$$

$$x^4 - 2x^2 - 16x + 1 = 0$$
$$or\ (x^2 + 3)^2 - 8(x + 1)^2 = 0\ or\ x^2 + 3 = \pm 2\sqrt{2}(x + 1)$$

$$Now\ x^2 + 3 = 2\sqrt{2}(x + 1)\left(\ as\ x^2 + 3 = -2\sqrt{2}(x + 1)\ does\ not\ has\ real\ roots\right)$$
$$or\ x^2 - 2\sqrt{2}x + \left(3 - 2\sqrt{2}\right) = 0\ or\ x = \frac{2\sqrt{2} \pm \sqrt{8 - 4\left(3 - 2\sqrt{2}\right)}}{2} = \sqrt{2} \pm \sqrt{2\sqrt{2} - 1}$$
$$Clearly\ roots\ from\ x^4 - 16x + 4 = 0\ does\ not\ satify\ given\ equation$$
$$since\ x^4 - 16x + 4 = 0 \Rightarrow x^4 = 16x - 4$$

$$If\ we\ put\ value\ of\ \ x^4 in\ given\ equation\ LHS\ part\ then$$
$$LHS = x^4 - x^2 - 16x + 2 = (16x - 4) - x^2 - 16x + 2 = -2 - x^2 < 0\ but$$
$$RHS = 2\sqrt{x^2 + 4x} > 0$$
$$For\ this\ required\ solution\ x = \sqrt{2} \pm \sqrt{2\sqrt{2} - 1}$$