ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt[3]{x+2024}+\sqrt[3]{x+2028}=\sqrt[3]{x+2025}+\sqrt[3]{x+2027}$$

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$$Let\ a=\sqrt[3]{x+2024}, b=\sqrt[3]{x+2025}, c=\sqrt[3]{x+2027}, d=\sqrt[3]{x+2028}$$

$$then\ \sqrt[3]{x+2024}+\sqrt[3]{x+2028}=\sqrt[3]{x+2025}+\sqrt[3]{x+2027}\Rightarrow a+d=b+c$$

$$(a+d)^3=a^3+d^3+3ad(a+d)=x+2024+x+2028+3ad(a+d)=$$
$$=2x+4052+3ad(a+d)$$

$$(b+c)^3=b^3+c^3+3bc(b+c)=x+2025+x+2027+3bc(b+c)=$$
$$=2x+4052+3bc(b+c)$$

$$Since\ a+d=b+c\ then\ (a+d)^3=(b+c)^3$$
$$or\ 2x+4052+3ad(a+d)=2x+4052+3bc(b+c)$$
$$or\ ad(a+d)=bc(b+c)\ or,\quad ad(a+d)\overset{a+d=b+c}{=}bc(a+d)$$
$$or\ (a+d)(ad-bc)=0$$

$$If a+d=0\Rightarrow a^3=-d^3\ or\ x+2024=-(x+2028)\ or$$
$$2x=-4052\ or\ x=-2026$$

$$Now\ ad-bc\neq 0$$
$$because\ a^3d^3=(x+2024)(x+2028)=x^2+x(4052)+2028\times 2024$$
$$b^3c^3=(x+2025)(x+2027)=x^2+x(4052)+2025\times 2027$$

$$Clearly\ a^3d^3\neq b^3c^3\ as\ 2028\times 2024\neq 2025\times 2027\ so\ ad\neq bc\ or\ ad-bc\neq 0$$
$$Required\ solution\ x=-2026$$