ROMANIAN MATHEMATICAL MAGAZINE

Solve for reals:

$$x(x+4)=16+2x\sqrt{x(x-4)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x(x+4)=16+2x\sqrt{x(x-4)}\ \ clearly\ x(x-4)\geq 0\ or\ x\leq 0\ or, x\geq 4$$

$$Let\ x=2t\ then\ given\ equation\ can\ be\ written\ as:$$

$$2t(2t+4)=16+2.2t\sqrt{2t(2t-4)}\ or\ 4t^2+8t=16+8t\sqrt{t(t-2)}$$
$$t^2+2t=4+2t\sqrt{t(t-2)}\ or\ \frac{t^2+2t-4}{2t}=\sqrt{t(t-2)}$$
$$or\left(\frac{t^2+2t-4}{2t}\right)^2=\left(\sqrt{t(t-2)}\,\right)^2$$
$$(L.H.S>0\ as\ t\leq 0\ or\ t\geq 2\ since\ x=2t\ \&\ \ x\leq 0\ or\ x\geq 4)$$

$$(t^2+2t-4)^2\geq 4t^2(t^2-2t)$$

$$t^4+4t^3-4t^2-16t+16=4t^4-8t^3$$

$$3t^4-12t^3+4t^2+16t-16=0$$

$$t^4-4t^3+\frac{4}{3}t^2+\frac{16}{3}t-\frac{16}{3}=0$$

$$Standard\ \ form\ t^4+pt^3+qt^2+rt+s=0$$
$$Let\ t=y=y-\frac{p}{4}=y-\frac{-4}{4}=y+1\ then:$$
$$(y+1)^4-4(y+1)^3+\frac{4}{3}(y+1)^2+\frac{16}{3}(y+1)-\frac{16}{3}=0$$

$$(y^4+4y^3+6y^2+4y+1)-4(y^3+3y^2+3y+1)+\frac{4}{3}(y^2+2y+1)+\frac{16}{3}(y+1)-\frac{16}{3}=0$$
$$y^4-\frac{14}{3}y^2-\frac{5}{3}=0\ or\ 3y^4-14y^2-5\geq 0\ or\ (y^2-5)(3y^2+1)=0$$
$$or\ y^2=5\ \left(as\ y^2\neq -\frac{1}{3}\right)\ or\ y=\pm\sqrt{5}$$
$$Solutions\ \ x=2t=2(y+1)=2\left(1\pm\sqrt{5}\,\right)$$