ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt{1+\frac{3}{x}}=\sqrt{1+4x}+2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1+4x\geq 0\ or, x\geq -\frac{1}{4}\ and\ 1+\frac{3}{x}\geq 0\ or, \frac{x+3}{x}\geq 0\ or, x\leq -3\ or\ x\geq 0$$

$$Combining\ both\ condition\ gives\ x>0$$

$$\sqrt{1+\frac{3}{x}}=\sqrt{1+4x}+2\ or\ \left(\sqrt{1+\frac{3}{x}}\right)^2=\left(\sqrt{1+4x}+2\right)^2$$

$$or, 1+\frac{3}{x}=1+4x+4+4\sqrt{1+4x}\ or, 4x^2+4x\sqrt{1+4x}=3$$

$$or, \frac{u^2-1}{4}(u^2-1)+(u^2-1)(u+1)\overset{1+4x=u^2}{=}3\ or, u^4+4u^3+2u^2-4u-3=12$$

$$or, u^4+4u^3+2u^2-4u-3=12\ or, (u^2+2u-3)(u^2+2u+1)=12$$

$$or, (y-3)(y-1)\overset{y=u^2+2u}{=}12\ or, y^2-2y-15=0\ or,$$

$$(y-5)(y+3)=0\ or, y=5\ ,-3$$

$$y=5\Rightarrow u^2+2u=5\ or, u=\frac{-2+\sqrt{24}}{2}=\sqrt{6}-1\ since\ u=\sqrt{1+4x}>1\ as\ x>0$$

$$y=-3\ or, u^2+2u+3=0\ not\ valid\ as\ D<0\ no\ real\ root$$

$$so, u=\sqrt{1+4x}=\left(\sqrt{6}-1\right)\ or, x=\frac{\left(\sqrt{6}-1\right)^2-1}{4}=\frac{3-\sqrt{6}}{2}$$