

ROMANIAN MATHEMATICAL MAGAZINE

Solve for reals :

$$\sqrt[3]{(x-1)^2} - \sqrt[3]{2(x-5)^2} + \sqrt[3]{(x-7)^2} = \sqrt[3]{4x}$$

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$$\text{Let : } \begin{cases} u^3 = x^2 - 2x + 1 \\ v^3 = x^2 - 14x + 49 \\ t^3 = 4x \end{cases} \rightarrow u^3 + v^3 - t^3 = 2x^2 - 20x + 50$$

Then :

$$u - \sqrt[3]{u^3 + v^3 - t^3} + v = t, \quad (u + v - t)^3 = u^3 + v^3 - t^3$$

$$u^3 + v^3 - t^3 + 3(u + v)(u - t)(v - t) = u^3 + v^3 - t^3$$

$$(u + v)(u - t)(v - t) = 0$$

$$\text{1) case : } \begin{cases} u + v = 0 \\ u^3 = -v^3 \\ x^2 - 2x + 1 = -x^2 + 14x - 49, \\ x^2 - 8x + 25 = 0 \\ x_{1,2} = 4 \pm 3i \end{cases}, \quad \text{2) case : } \begin{cases} u - t = 0 \\ u^3 = t^3 \\ x^2 - 2x + 1 = 4x \\ x^2 - 6x + 1 = 0 \\ x_{3,4} = 3 \pm 2\sqrt{2} \end{cases}$$

$$\text{3) case : } \begin{cases} v - t = 0 \\ v^3 = t^3 \\ x^2 - 14x + 49 = 4x \\ x^2 - 18x + 49 = 0 \\ x_{5,6} = 9 \pm 4\sqrt{2} \end{cases}$$

$$\text{Answer : } \{3 \pm 2\sqrt{2} ; 9 \pm 4\sqrt{2}\}$$