ROMANIAN MATHEMATICAL MAGAZINE

Solve for reals :

$$\sqrt{(1+x)^3}+\sqrt{(1-x)^3}+x^2=2$$

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$$\begin{cases} 1+x\geq 0 & \rightarrow \quad x\geq -1 \\ 1-x\geq 0 & \rightarrow \quad x\leq -1 \end{cases} \quad x\in[-1;1]$$

$$t=\sqrt{1+x},\ \ t\geq 0 \quad v=\sqrt{1-x}, \qquad v\geq 0$$

$$\sqrt{(1+x)^3}=t^3 \ \ and \ \ \sqrt{(1-x)^3}=v^3$$

$$\therefore t^2v^2=(1+x)(1-x)=1-x^2, \qquad x^2=1-t^2v^2$$

$$t^3+v^3+x^2=2, \qquad t^3+v^3+(1-t^2v^2)=2, \qquad t^3+v^3-t^2v^2-1=0$$

$$\begin{cases} s=t+v\,,\ \ p=tv \\ t^2+v^2=2,\ \ (t+v)^2-2tv=2,\ \ s^2-2p=2\,,\ \ p=\dfrac{s^2-2}{2} \end{cases}$$

$$\therefore t^3+v^3=(t+v)(t^2+v^2-tv)=s(2-p)$$

$$s(2-p)-p^2-1=0, \qquad 2s-sp-p^2-1=0$$

$$2s-s\left(\frac{s^2-2}{2}\right)-\left(\frac{s^2-2}{2}\right)^2-1=0, \qquad 8s-2s^3+4s-s^4+4s^2-8=0$$

$$4s^2-2s^3-s^4+12s-8=0, \qquad s^4+2s^3-4s^2-12s+8=0$$

$$(s-2)(s^3+4s^2+4s-4)=0$$

$$\begin{cases} s-2=0,\ \ s=2 \\ s^3+4s^2+4s-4=0 \ \ \textit{The equation has no real roots.} \end{cases}$$

$$s=2,\ \ t^2+v^2=2$$

$$\therefore \textit{The only positive numbers here are} : t=1 \ \ and \ \ v=1$$

$$\begin{cases} \sqrt{1+x}=1\rightarrow \quad 1+x=1 \ \rightarrow \ \ x=0 \\ \sqrt{1-x}=1\rightarrow \quad 1-x=1 \ \rightarrow \ \ x=0 \end{cases}$$