ROMANIAN MATHEMATICAL MAGAZINE

Solve for reals:

$$(x^6)^{x^6} = 6x - 5$$

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$$As\ L.H.S > 0\ so\ R.H.S > 0\ or\ 6x - 5 > 0\ or\ x > \frac{5}{6}$$

$$Let\ f(x) = (x^6)^{x^6} - (6x - 5\ ) = x^{6x^6} - (6x - 5)$$

$$f'(x) = 6x^5x^{6x^6}(6\ln x + 1) - 6 = 6\left(x^5x^{6x^6}(6\ln x + 1) - 1\right)$$

$$Case1: x \in \left(\frac{5}{6}, 1\right)\ then\ x^5x^{6x^6} < 1, 6\ln x < 1\ for\ this\ f'(x) < 0$$

$$so\ f(x) is\ decreasing\ on\ \left(\frac{5}{6}, 1\right)$$

$$Case2: x = 1\ then\ f(1) = 1 - (6 - 5) = 0\ so\ x = 1\ is\ root\ \ of f(x) = 0$$

$$Case\ 3: x > 1\ then \ln x > 0, x^5x^{6x^6} > 1\ for\ this\ f'(x) > 0$$

$$so\ f(x) is\ increasing\ for\ x > 1$$

$$Conclusion: f(x)\ is\ decreasing\ on\ \left(\frac{5}{6}, 1\right)\ \&\ increasing\ for\ x > 1\ and$$

$$at\ x = 1\ f(x) = 0$$

$$so\ required\ solution\ x = 1.$$