

ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt{(1+x)^3} + \sqrt{(1-x)^3} - 2(\sqrt{1+x} + \sqrt{1-x}) = -2$$

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$$\text{Denote: } \begin{cases} a = \sqrt{1+x} \\ b = \sqrt{1-x} \end{cases} \Rightarrow \begin{cases} a^3 + b^3 - 2(a+b) = -2 \\ a^2 + b^2 = 2 \end{cases}$$

$$\text{Denote: } \begin{cases} S = a+b \\ P = ab \end{cases} \Rightarrow \begin{cases} S^3 - 3SP - 2P = -2 \\ S^2 - 2P = 2 \end{cases}$$

$$S^2 - 2P = 2 \Rightarrow S^2 - 2 = 2P \Rightarrow P = \frac{S^2 - 2}{2}$$

$$S^3 - 3S \cdot \frac{S^2 - 2}{2} - 2S = -2, \quad 2S^3 - 3S^3 + 6S - 4S + 4 = 0$$

$$-S^3 + 2S + 4 = 0 \Rightarrow S^3 - 2S - 4 = 0$$

$$S^3 - 2S^2 + 2S^2 - 4S + 2S - 4 = 0, \quad S^2(S-2) + 2S(S-2) + 2(S-2) = 0$$

$$(S-2)(S^2 + 2S + 2) = 0, \quad S^2 + 2S + 2 = 0 \Rightarrow \Delta < 0$$

$$S - 2 = 0 \Rightarrow S = 2 \Rightarrow P = 1 \Rightarrow a = b = 1 \Rightarrow \sqrt{1+x} = \sqrt{1-x} = 1 \Rightarrow x = 0$$