ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt{(1+x)^3}+\sqrt{(1-x)^3}-\sqrt{1-x^2}=1$$

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Denote: $\begin{cases} a=\sqrt{1+x} \\ b=\sqrt{1-x} \end{cases}$; $\begin{cases} S=a+b \\ P=ab \end{cases}$; $a,b>0\Rightarrow S>0$

$$\begin{cases} a^3+b^3-ab=1 \\ a^2+b^2=2 \end{cases}\Rightarrow\begin{cases} S^3-3SP-P=1 \\ S^2-2P=2 \end{cases}, \quad S^2-2=2P\Rightarrow P=\frac{S^2-2}{2}$$

$$S^3-3S\cdot\frac{S^2-2}{2}-\frac{S^2-2}{2}=1, \quad 2S^3-3S^3+6S-S^2+2=2$$

$$-S^3-S^2+6S=0\Rightarrow -S(S^2+S-6)=0,\ S^2+S-6=0$$

$$\Rightarrow\begin{cases} S_1=2\Rightarrow P_1=1 \\ S_2=-3\Rightarrow P_2=\frac{9-2}{2}=\frac{7}{2} \end{cases}$$

$$z^2-S_2z+P_2=0\Rightarrow z^2+3z+\frac{7}{2}=0\Rightarrow\Delta<0$$

$$z^2-S_1z+P_1=0\Rightarrow z^2-2z+1=0\Rightarrow z_1=z_2=1$$

$$\Rightarrow a=b=1\Rightarrow\sqrt{1+x}=\sqrt{1-x}=1\Rightarrow x=0$$