

ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt[3]{1+x} + \sqrt[3]{1-x} - \sqrt[3]{1-x^2} = 1$$

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Denote:

$$\begin{cases} \sqrt[3]{1+x} = a \\ \sqrt[3]{1-x} = b \end{cases} \Rightarrow \begin{cases} a^3 = 1+x \\ b^3 = 1-x \end{cases} \Rightarrow a^3 + b^3 = 2$$

$$\sqrt[3]{1+x} + \sqrt[3]{1-x} - \sqrt[3]{1-x^2} = 1 \Rightarrow a + b - ab = 1$$

Denote: $S = a + b$; $P = ab$

$$\begin{cases} a^3 + b^3 = 2 \\ a + b - ab = 1 \end{cases} \Rightarrow \begin{cases} S^3 - 3SP = 2 \\ S - P = 1 \end{cases}$$

$$P = S - 1 \Rightarrow S^3 - 3S(S - 1) = 2$$

$$S^3 - 3S^2 + 3S - 2 = 0$$

$$S^3 - 2S^2 - S^2 + 2S + S - 2 = 0$$

$$S^2(S - 2) - S(S - 2) + (S - 2) = 0$$

$$(S - 2)(S^2 - S + 1) = 0$$

$$S - 2 = 0 \Rightarrow S = 2 \Rightarrow P = 2 - 1 = 1$$

$$\begin{cases} S = 2 \\ P = 1 \end{cases} \Rightarrow a = b = 1 \Rightarrow \begin{cases} \sqrt[3]{1+x} = 1 \\ \sqrt[3]{1-x} = 1 \end{cases} \Rightarrow x = 0$$