

ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt{1+x} + \sqrt{1-x} + 2x^2 = 2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{Denote: } \begin{cases} \sqrt{1+x} = a \\ \sqrt{1-x} = b \end{cases} \Rightarrow \begin{cases} 1+x = a^2 \\ 1-x = b^2 \end{cases} \Rightarrow a^2 + b^2 = 2$$

$$2 - 2x^2 = 2(1-x^2) = 2(1-x)(1+x) = 2a^2b^2$$

$$\begin{cases} a^2 + b^2 = 2 \\ a + b = 2a^2b^2 \end{cases} \text{ Denote: } \begin{cases} S = a + b \\ P = ab \end{cases} \Rightarrow \begin{cases} S^2 - 2P = 2 \\ S = 2P^2 \end{cases} \Rightarrow (2P^2)^2 - 2P - 2 = 0$$

$$4P^4 - 4P + 2P - 2 = 0 \Rightarrow 4P(P^3 - 1) + 2(P - 1) = 0$$

$$4P(P - 1)(P^2 + P + 1) + 2(P - 1) = 0$$

$$2(P - 1)(2P^2 + 2P^2 + 2P + 1) = 0$$

$$a, b > 0 \Rightarrow ab > 0 \Rightarrow P > 0 \Rightarrow 2P^3 + 2P^2 + 2P + 1 > 0$$

$$\Rightarrow P - 1 = 0 \Rightarrow P = 1 \Rightarrow S = 2 \cdot 1^2 = 2$$

$$\begin{cases} S = 2 \\ P = 1 \end{cases} \Rightarrow a = b = 1 \Rightarrow 1 - x = 1 + x = 1 \Rightarrow x = 0$$