

ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt[3]{1+x} + \sqrt[3]{1-x} = 2\sqrt[3]{1-x^2}$$

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Denote:

$$\begin{cases} \sqrt[3]{1+x} = a \\ \sqrt[3]{1-x} = b \end{cases} \Rightarrow a^3 + b^3 = 2, \quad \begin{cases} a^3 + b^3 = 2 \\ a + b = 2ab \end{cases}$$

Denote:

$$\begin{aligned} \begin{cases} S = a + b \\ P = ab \end{cases} &\Rightarrow \begin{cases} S^3 - 3SP = 2 \\ S = 2P \end{cases} \Rightarrow (2P)^3 - 3 \cdot 2P \cdot P = 2 \\ 8P^3 - 6P^2 - 2 &= 0 \Rightarrow 4P^3 - 3P^2 - 1 = 0 \\ 4P^3 - 4P^2 + P^2 - 1 &= 0 \Rightarrow 4P^2(P-1) + (P-1)(P+1) = 0 \\ (P-1)(4P^2 + P + 1) &= 0 \\ a, b > 0 &\Rightarrow P = ab > 0 \Rightarrow 4P^2 + P + 1 > 0 \\ \Rightarrow P - 1 = 0 &\Rightarrow P = 1 \Rightarrow S = 2 \cdot 1 \Rightarrow S = 2 \\ \begin{cases} S = 2 \\ P = 1 \end{cases} &\Rightarrow a = b = 1 \Rightarrow 1 + x = 1 - x = 1 \Rightarrow x = 0 \end{aligned}$$