ROMANIAN MATHEMATICAL MAGAZINE

Solve for reals:

$$x\sqrt{x+7}+\sqrt{9-x}=4\sqrt{x^2+1}$$

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$$Set\ p=\sqrt{x+7},q=\sqrt{9-x},r=\sqrt{x^2+1}$$
$$Clearly,p^2+q^2=x+7+9-x=16=4^2$$
$$Let\ \ p=4\cos t\,,q=4\sin t\,,t\in\left[0,\frac{\pi}{2}\right]\ and\ \ p=\sqrt{x+7}\ or$$
$$x=p^2-7=16\cos^2 t-7\ (1)$$

$$Using\ above\ substitution\ we\ get:$$
$$x\sqrt{x+7}+\sqrt{9-x}=4\sqrt{x^2+1}\ \ or$$
$$(16\cos^2 t-7\ )\cos t+\sin t=\sqrt{(16\cos^2 t-7\ )^2+1}$$
$$or\ A\cos t+\sin t\overset{16\cos^2 t-7=A}{=}\sqrt{A^2+1}$$
$$A\cos t+\sin t\overset{Cauchy\ Schwarz}{\leq}\sqrt{(A^2+1)(\cos^2 t+\sin^2 t)}=\sqrt{A^2+1}$$
$$for\ equality\ we\ take\cos t=\lambda A,\sin t=\lambda\ then\cos^2 t+\sin^2 t=1\Rightarrow\ \lambda^2=\frac{1}{A^2+1}$$
$$\cos^2 t=\lambda^2A^2=\frac{A^2}{A^2+1}\ ,from\ (1)\ we\ get:$$
$$x=p^2-7=16\cos^2 t-7\ \ or\ \ \cos^2 t=\frac{A+7}{16}$$
$$\frac{A^2}{A^2+1}=\frac{A+7}{16}\ or,A^3-9A^2+A+7=0\ or(A-1)(A^2-8A-7)=0$$
$$or\ A=1\ and\ (A^2-8A-7)=0\Rightarrow A=\frac{8\pm\sqrt{64+28}}{2}or,A=4\pm\sqrt{23}$$
$$note\ that\ A=(16\cos^2 t-7\ )=x\ by\ (1)$$

$$but\ A=x=\ 4-\ \sqrt{23}\ not\ a\ valid\ solution\ as\sqrt{x+7}\ not\ valid\ at\ x=\ 4-\ \sqrt{23}$$
$$so\ required\ solution\ x=4,4+\sqrt{23}\ \ in\ real$$