Solve fore real numbers:

$$\sqrt[4]{x-\sqrt{x^2-1}}+\sqrt{x+\sqrt{x^2-1}}=2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Denote:

$$a=\sqrt[4]{x-\sqrt{x^2-1}};\ b=\sqrt[4]{x+\sqrt{x^2-1}}$$

$$a\cdot b=\sqrt[4]{x^2-\left(\sqrt{x^2-1}\right)^2}=\sqrt[4]{x^2-x^2+1}=1$$

$$b=\frac{1}{a};\ a+b^2=2\Rightarrow a+\frac{1}{a^2}=2$$

$$a^3+1-2a^2=0\Rightarrow a^3-2a^2+1=0$$

$$a^3-a^2-(a^2-1)=0\Rightarrow a^2(a-1)(a-1)(a+1)=0$$

$$(a-1)(a^2-a-1)=0$$

$$a-1=0\Rightarrow a_1=1\Rightarrow \sqrt[4]{x-\sqrt{x^2-1}}=1$$

$$x-\sqrt{x^2-1}=1\Rightarrow x-1=\sqrt{x^2-1}\Rightarrow x^2-2x+1=x^2-1$$

$$-2x=-2\Rightarrow x=1$$

$$a^2-a-1=0\Rightarrow a_{2,3}=\frac{2\pm\sqrt{5}}{2}$$

$$\sqrt[4]{x-\sqrt{x^2-1}}=a\Rightarrow x-\sqrt{x^2-1}=a^2$$

$$x-a^2=\sqrt{x^2-1}\Rightarrow x^2-2xa^2+a^4=x^2-1$$

$$2xa^2=a^4+1\Rightarrow x=\frac{a^4+1}{2a^2}\Rightarrow x=\frac{a^2}{2}+\frac{1}{2a^2}$$

$$x_1=1;\ x_{2,3}=\frac{a^2}{2}+\frac{1}{2a^2};\ a\in\left\{\frac{2\pm\sqrt{5}}{2}\right\}$$