

ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt[5]{1+x} + \sqrt[5]{1-x} = 2$$

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Denote:

$$a = \sqrt[5]{1+x}, \quad b = \sqrt[5]{1-x}, \quad S = a + b, \quad P = ab$$

$$a + b = 2 \Rightarrow S = 2$$

$$a^5 + b^5 = 1 + x + 1 - x = 2$$

$$(a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) = 2$$

$$2(a^4 + b^4 - ab(a^2 + b^2) + a^2b^2) = 2$$

$$a^4 + b^4 - ab(a^2 + b^2) + a^2b^2 = 1$$

$$(a^2 + b^2)^2 - 2a^2b^2 - ab(a^2 + b^2) + a^2b^2 = 1$$

$$(S^2 - 2P)^2 - P(S^2 - 2P) - P^2 = 1$$

$$(2^2 - 2P)^2 - P(2^2 - 2P) - P^2 = 1$$

$$16 - 16P + 4P^2 - 4P + 2P^2 - P^2 = 1$$

$$5P^2 - 20P + 15 = 0$$

$$P^2 - 4P + 3 = 0 \Rightarrow (P - 1)(P - 3) = 0$$

$$P = 3, S = 2 \Rightarrow a, b \notin \mathbb{R}$$

$$P = 1, S = 2 \Rightarrow a = b = 1 \Rightarrow x = 0$$