ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$x^2 - x + 1 = \sqrt[3]{3x-2}$$

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$$Let\ t = \sqrt[3]{3x-2}\ \ or, t^3 = 3x - 2\ or, x = \frac{t^3+2}{3}\ (1)$$

$$x^2 - x + 1 = \sqrt[3]{3x-2}$$
$$\left(\frac{t^3+2}{3}\right)^2 - \frac{t^3+2}{3} + 1 = t$$
$$(t^3+2)^2 - 3(t^3+2) + 9 = 9t$$
$$t^6 + 4t^3 + 4 - 3t^3 - 6 + 9 = 9t$$
$$t^6 + t^3 - 9t + 7 = 0$$
$$(t-1)^2(t^4 + 2t^3 + 3t^2 + 5t + 7) = 0\ (2)$$

$$(t^4 + 2t^3 + 3t^2 + 5t + 7) = t^2(1+t)^2 + (2t^2 + 5t + 7)$$
$$2t^2 + 5t + 7 > 0\ as\ a = 2 > 0\ and\ D = 5^2 - 4.2.7 = 55 - 56 < 0$$
$$and\ \ t^2(1+t)^2 \geq 0\ \forall t \in R, so\ t^2(1+t)^2 + (2t^2 + 5t + 7) > 0$$

$$For\ this\ from (2) we\ get\ (t-1)^2 = 0\ or\ t = 1$$
$$so\ required\ solution\ x = \frac{t^3+2}{3} = \frac{1^3+2}{3} = 1$$