ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt[3]{x+6}+\sqrt{x-1}=x^2-1$$

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$$\textit{Since } \sqrt{x-1} \textit{ is define on } x \ge 1\,,$$
$$\textit{so we are going to solve above equation} \in [1,\infty)$$

$$\textit{Let } f(x)=\sqrt[3]{x+6}+\sqrt{x-1}-(x^2-1)\,,x\ge 1$$
$$f(1)=\sqrt[3]{7}+0-0=\sqrt[3]{7}>0\,,f(2)=\sqrt[3]{8}+1-3=0$$
$$\textit{so } x=2 \textit{ is a solution}$$

$$\textit{Now we check nature of the function in } x\in[1,2)$$
$$f'(x)=\frac{1}{3}(x+6)^{-\frac{2}{3}}+\frac{1}{2\sqrt{x-1}}-2x$$
$$f''(x)=-\frac{2}{9}(x+6)^{-\frac{5}{3}}-\frac{1}{4}(x-1)^{-\frac{3}{2}}-2=-\left(\frac{2}{9}(x+6)^{-\frac{5}{3}}+\frac{1}{4}(x-1)^{-\frac{3}{2}}+2\right)<0$$
$$\textit{so } f \textit{ is concave in } [1,2], \textit{for a concave function we have standard inequality}$$
$$\textit{for } x\in[1,2)\Rightarrow x=(2-x).1+(x-1).2$$
$$\textit{then } f\big((2-x).1+(x-1).2\,\big)\ge(2-x)f(1)+(x-1)f(2)$$
$$\textit{or } f(x)\ge(2-x)f(1)>0 \textit{ as } x\in[1,2)\ \&\ f(1)=\sqrt[3]{7}>0$$
$$\textit{so } f(x)>0 \textit{ in } [1,2) \textit{ and } x=2\ f(2)=0$$

$$\textit{Now we check nature of the function in } x>2$$
$$f'(x)=\frac{1}{3}(x+6)^{-\frac{2}{3}}+\frac{1}{2\sqrt{x-1}}-2x$$
$$\textit{as } x>2 \textit{ then} \frac{1}{3}(x+6)^{-\frac{2}{3}}<\frac{1}{3}.8^{-\frac{2}{3}}=\frac{1}{12}\,,\frac{1}{2\sqrt{x-1}}<\frac{1}{2.1}=\frac{1}{2},-2x<-4$$
$$\textit{for this } f'(x)<\frac{1}{12}+\frac{1}{2}-4=-\frac{41}{12}<0$$
$$\textit{which indicate that } f(x)\ \textit{ is decreasing } \forall x>2$$
$$\textit{that's why after hitting } 0 \textit{ at } x=2$$
$$\textit{the function goes negative and never come back up}$$
$$\textit{so requested soution } x=2$$