

ROMANIAN MATHEMATICAL MAGAZINE

Solve for real numbers:

$$\sqrt{1+x} + \sqrt{1-x} = 2(1-x^2)$$

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Denote: $\sqrt{1+x} = a > 0; \sqrt{1-x} = b > 0 \Rightarrow P = ab > 0; S = a + b$

$$\begin{aligned}\sqrt{1+x} = a &\Rightarrow 1+x = a^2 \\ \sqrt{1-x} = b &\Rightarrow 1-x = b^2\end{aligned} \Rightarrow a^2 + b^2 = 2 \Rightarrow S^2 - 2P = 2$$

$$\sqrt{1+x} + \sqrt{1-x} = 2(1-x)(1+x) \Rightarrow a + b = 2a^2b^2$$

$$S = 2P^2 \Rightarrow (2P^2)^2 - 2P = 2$$

$$4P^4 - 2P - 2 = 0 \Rightarrow 2P^4 - P - 1 = 0$$

$$2P^4 - 2P + P - 1 = 0 \Rightarrow 2P(P^3 - 1) + (P - 1) = 0$$

$$2P(P - 1)(P^2 + P + 1) + (P - 1) = 0$$

$$(P - 1)(2P^3 + 2P^2 + 2P + 1) = 0$$

$$P > 0 \Rightarrow 2P^3 + 2P^2 + 2P + 1 \neq 0$$

$$P - 1 = 0 \Rightarrow P = 1 \Rightarrow S = 2 \Rightarrow a = b = 1$$

$$\sqrt{1+x} = 1 \Rightarrow 1+x = 1 \Rightarrow x = 0$$