ROMANIAN MATHEMATICAL MAGAZINE

Solve for $x \in \left(0, \frac{\pi}{2}\right)$:

$$1+\frac{1}{2}\cos x+\frac{1}{4}\cos 2x+\frac{1}{8}\cos 3x+\frac{1}{16}\cos 4x+\cdots=\frac{23+3\sqrt{5}}{22}$$

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$$We\ know\ that\ e^{i\theta}=\cos\theta+i\sin\theta\ and\ \cos\theta=\frac{\left(e^{i\theta}+e^{-i\theta}\right)}{2}$$

$$Componendo\ dividendo\ if\ \frac{a}{b}=\frac{c}{d}\ then\ \frac{a+b}{a-b}=\frac{c+d}{c-d}$$

$$\cos\frac{\pi}{5}=\frac{\sqrt{5}+1}{4}$$

$$1+\frac{1}{2}\cos x+\frac{1}{4}\cos 2x+\frac{1}{8}\cos 3x+\frac{1}{16}\cos 4x+\cdots=$$

$$=\left(\frac{1}{2}+\frac{1}{2}\right)+\frac{1}{2}\frac{e^{ix}+e^{-ix}}{2}+\frac{1}{4}\frac{e^{i2x}+e^{-i2x}}{2}+\frac{1}{8}\frac{e^{i3x}+e^{-i3x}}{2}+\frac{1}{16}\frac{e^{i4x}+e^{-i4x}}{2}\ \ldots=$$

$$=\frac{1}{2}\left(1+\frac{1}{2}e^{ix}+\frac{1}{4}e^{2ix}+\frac{1}{8}e^{3ix}+\frac{1}{16}e^{4ix}+\cdots\right)+$$

$$+\frac{1}{2}\left(1+\frac{1}{2}e^{-ix}+\frac{1}{4}e^{-2ix}+\frac{1}{8}e^{-3ix}+\frac{1}{16}e^{-4ix}+\cdots\right)$$

$$=\frac{1}{2}\frac{1}{1-\frac{1}{2}e^{ix}}+\frac{1}{2}\frac{1}{1-\frac{1}{2}e^{-ix}}\ (infinite\ G.P\ series)$$

$$=\frac{1}{2-e^{ix}}+\frac{1}{2-e^{-ix}}=\frac{4-(e^{ix}+e^{-ix})}{(2-e^{ix})(2-e^{-ix})}=$$

$$=\frac{4-2\cos x}{(2-\cos x+i\sin x)(2-\cos x+i\sin x)}=$$

$$=\frac{4-2\cos x}{(2-\cos x)^2+\sin^2 x}=\frac{4-2\cos x}{5-4\cos x}$$

$$Now\ by\ the\ problem:\ \frac{4-2\cos x}{5-4\cos x}=\frac{23+3\sqrt{5}}{22}$$

$$\frac{8-4\cos x}{5-4\cos x}=\frac{23+3\sqrt{5}}{11}$$

$$\frac{13-8\cos x}{3}=\frac{(34+3\sqrt{5})}{12+3\sqrt{5}}\ (using\ Componendo\ dividendo)$$

$$or\ 13-8\cos x=\frac{(34+3\sqrt{5})}{4+\sqrt{5}}\ or\ 8\cos x=13-\frac{(34+3\sqrt{5})}{4+\sqrt{5}}$$

$$or\ 8\cos x=\frac{18+10\sqrt{5}}{4+\sqrt{5}}\ or\ 4\cos x=\frac{9+4\sqrt{5}}{4+\sqrt{5}}=\frac{(4+\sqrt{5})(\sqrt{5}+1)}{4+\sqrt{5}}$$

$$or\ \cos x=\frac{(\sqrt{5}+1)}{4}=\cos\frac{\pi}{5}\ or\ x=\frac{\pi}{5}$$