ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda \geq 0$ fixed, then solve for reals:

$$x^2 - 3x + 1 = -\frac{1}{\sqrt{\lambda + 2}}\sqrt{x^4 + \lambda x^2 + 1}$$

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$$\frac{1}{\sqrt{\lambda+2}}\sqrt{x^4+\lambda x^2+1} > 0\ \forall\ x \in R\ then\ R.H.S\ \frac{-1}{\sqrt{\lambda+2}}\sqrt{x^4+\lambda x^2+1} < 0$$

$$so\ L.H.S\ must\ be < 0\ or\ x^2 - 3x + 1 < 0\ (1)$$

$$x^2 - 3x + 1 = -\frac{1}{\sqrt{\lambda+2}}\sqrt{x^4+\lambda x^2+1}$$

$$(x^2 - 3x + 1)^2 = \frac{1}{\lambda+2}(x^4+\lambda x^2+1)$$

$$(\lambda+2)(x^2-3x+1)^2 - (x^4+\lambda x^2+1) = 0$$

$$(\lambda+2)(x^4+9x^2+1-6x^3-6x+2x^2) - (x^4+\lambda x^2+1) = 0$$

$$(\lambda+1)x^4 - 6(\lambda+2)x^3 + (10\lambda+22)x^2 - 6(\lambda+2)x + (\lambda+1) = 0$$

$$(x-1)^2\left((\lambda+1)x^2 - (4\lambda+10)x + (\lambda+1)\right) = 0$$

$$now, x - 1 = 0 \Rightarrow x = 1$$

$$and\ \left((\lambda+1)x^2 - (4\lambda+10)x + (\lambda+1)\right) = 0$$

$$\Rightarrow\ x^2 - \frac{4\lambda+10}{\lambda+1}x + 1 = 0\ \ (2)$$

$$Clearly\ product\ of\ the\ two\ roots = 1\ ,$$

$$which\ indicate\ that\ roots\ are\ reiprocal\ to\ each\ other\ let\ roots\ are\ r, \frac{1}{r}$$

$$then\ from\ (2), r + \frac{1}{r} = \frac{4\lambda+10}{\lambda+1} = 4 + \frac{6}{\lambda+1} > 4\ as\ \lambda \geq 0$$

$$but\ we\ should\ also\ remember\ that\ from (1)\ that\ roots\ of\ eqution\ (2) must\ satisfy$$

$$x^2 - 3x + 1 < 0$$

$$x^2 - 3x - 1 < 0\ or\ \left(x - \frac{3+\sqrt{5}}{2}\right)\left(x - \frac{3-\sqrt{5}}{2}\right) < 0\ so:$$

$$\frac{3-\sqrt{5}}{2} < x < \frac{3+\sqrt{5}}{2}\ which\ indicate\ x > 0$$

$$x + \frac{1}{x} - 3 = \frac{x^2-3x+1}{x} < 0\ as\ \ x^2 - 3x + 1 < 0\ and\ x > 0$$

$$or, x + \frac{1}{x} < 3\ but\ from (2) we\ get$$

$$r + \frac{1}{r} > 4\ , their\ does\ not\ exist\ any\ x\ in\ equation (2)$$

$$so\ the\ only\ solution: x = 1$$