ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda \geq 0$ then solve for reals:

$$x^2 + 2(x-\lambda)\sqrt{x-\lambda} + (1-2\lambda)x + \lambda(\lambda-1) = \lambda^2(\lambda+1)^2$$

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$$\textit{Domain}: x-\lambda \geq 0 \ or, x \geq \lambda$$
$$\textit{Let}\ t = \sqrt{x-\lambda}\ (t \geq 0) \Rightarrow\ x = t^2 + \lambda\ (1)$$
$$x^2 + 2(x-\lambda)\sqrt{x-\lambda} + (1-2\lambda)x + \lambda(\lambda-1) \overset{(1)}{=}$$
$$= (t^2+\lambda)^2 + 2t^3 + (1-2\lambda)(t^2+\lambda) + \lambda(\lambda-1) =$$
$$= (t^4 + 2t^2\lambda + \lambda^2) + 2t^3 + (t^2 + \lambda - 2\lambda t^2 - 2\lambda^2) + \lambda^2 - \lambda =$$
$$= t^4 + 2t^3 + t^2 \ \ = t^2(t+1)^2$$

$$x^2 + 2(x-\lambda)\sqrt{x-\lambda} + (1-2\lambda)x + \lambda(\lambda-1) = \lambda^2(\lambda+1)^2$$
$$t^2(t+1)^2 = \lambda^2(\lambda+1)^2 \ or, t(t+1) = \lambda(\lambda+1), (t,\lambda \geq 0)$$
$$t^2 + t - (\lambda^2+\lambda) = 0 \ or\ (t-\lambda)(t+\lambda+1) = 0 \Rightarrow\ t-\lambda = 0 \ or\ t = \lambda$$
$$\textit{Since}: t+\lambda+1 = 0 \Rightarrow t = -1-\lambda \ \textit{not possible}\ \ as\ t > 0\ and\ \lambda > 0$$

$$\textit{Required solution}\ x \overset{(1)}{=} \lambda + t^2 = \lambda + \lambda^2.$$