ROMANIAN MATHEMATICAL MAGAZINE

If $\lambda > 0$ then solve for reals:

$$\left(\frac{x+\lambda}{2}\right)^9+\left(\frac{x-\lambda}{2}\right)^9=\left(\frac{x^3+3\lambda^2x}{4}\right)^3$$

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$$Let\ A=\frac{x+\lambda}{2}, B=\frac{x-\lambda}{2}$$

$$A^3+B^3=\frac{1}{8}((x+\lambda)^3+(x-\lambda)^3)=\frac{1}{8}2x((x+\lambda)^2-(x+\lambda)(x-\lambda)+(x-\lambda)^2)=$$

$$=\frac{x}{4}(x^2+3\lambda^2)=\frac{(x^3+3\lambda^2x)}{4}\ (1)$$

$$Now: \left(\frac{x+\lambda}{2}\right)^9+\left(\frac{x-\lambda}{2}\right)^9=\left(\frac{x^3+3\lambda^2x}{4}\right)^3\Rightarrow$$

$$A^9+B^9=(A^3+B^3)^3\ or\ A^9+B^9-(A^3+B^3)^3=0\ (2)$$

$$We\ put\ u=A^3, v=B^3\ then\ A^9+B^9-(A^3+B^3)^3=u^3+v^3-(u+v)^3=$$

$$=u^3+v^3-\left(u^3+v^3+3uv(u+v)\right)=-3uv(u+v)=-3A^3B^3(A^3+B^3)\ (3)$$

$$AB=\frac{x+\lambda}{2}\times\frac{x-\lambda}{2}=\frac{x^2-\lambda^2}{4}\Rightarrow A^3B^3=\frac{1}{64}(x^2-\lambda^2)^3\ (4)$$

$$From\ (2) we\ have\ A^9+B^9-(A^3+B^3)^3=0$$

$$-A^3B^3(A^3+B^3)\overset{(3)}{=}0\ or\ -\frac{3}{64}(x^2-\lambda^2)^3\frac{(x^3+3\lambda^2x)}{4}\overset{(1)\&(4)}{=}0$$

$$x^2-\lambda^2=0\Rightarrow\ x=\pm\lambda\ and\ \frac{(x^3+3\lambda^2x)}{4}=0\ gives\ x(x^2+3\lambda^2)=0\ or\ x=0$$

$$Required\ solution\ x=\pm\lambda, 0.$$