

ROMANIAN MATHEMATICAL MAGAZINE

Let be $\lambda \geq 3$ fixed. Solve for reals:

$$x + \sqrt{x + \frac{5}{2} + \sqrt{x + \frac{9}{4}}} = \frac{\lambda^2 - 8}{4}$$

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$$x + \sqrt{x + \frac{9}{4} + 2 \cdot \frac{1}{2} \sqrt{x + \frac{9}{4} + \frac{1}{4}}} = \frac{\lambda^2 - 8}{4}$$

$$x + \sqrt{\left(\sqrt{x + \frac{9}{4} + \frac{1}{4}}\right)^2} = \frac{\lambda^2 - 8}{4}, \quad x + \sqrt{x + \frac{9}{4} + \frac{1}{2} - \frac{\lambda^2 - 8}{4}} = 0$$

Denote:

$$x + \frac{9}{4} = a^2 \Rightarrow x = a^2 - \frac{9}{4}$$

$$a^2 - \frac{9}{4} + \sqrt{a^2} + \frac{1}{2} - \frac{\lambda^2}{4} + 2 = 0, \quad a^2 + a + \frac{1}{2} + 2 - \frac{9}{4} - \frac{\lambda^2}{4} = 0$$

$$4a^2 + 4a + 2 + 8 - 9 - \lambda^2 = 0, \quad 4a^2 + 4a + 1 - \lambda^2 = 0$$

$$\Delta = 16 - 16(1 - \lambda^2) = 16\lambda^2; \sqrt{\Delta} = 4\lambda, \quad a = \frac{-4 + 4\lambda}{8} = \frac{\lambda - 1}{2}$$

$$x = \left(\frac{\lambda - 1}{2}\right)^2 - \frac{9}{4} = \frac{\lambda^2 - 2\lambda + 1}{4} - \frac{9}{4} = \frac{\lambda^2 - 2\lambda}{4} - 2$$