

ROMANIAN MATHEMATICAL MAGAZINE

Let be $\lambda \geq 0$ fixed. Solve for reals:

$$\sqrt{\lambda^2 + x} + \sqrt{\lambda^2 - x} + \sqrt{\lambda^4 - x^2} = \lambda^2 + 2\lambda$$

Proposed by Marin Chirciu – Romania

Solution by Daniel Sitaru – Romania

$$\text{Denote: } \begin{cases} a = \sqrt{\lambda^2 + x} \\ b = \sqrt{\lambda^2 - x} \end{cases} \Rightarrow \begin{cases} a^2 + b^2 = 2\lambda^2 \\ a + b + ab = \lambda^2 + 2\lambda \end{cases}$$

$$\text{Denote: } \begin{cases} S = a + b \\ P = ab \end{cases} \Rightarrow \begin{cases} S^2 - 2P = 2\lambda^2 \\ S + P = \lambda^2 + 2\lambda \end{cases}$$

$$P = \lambda^2 + 2\lambda - S \Rightarrow S^2 - 2(\lambda^2 + 2\lambda - S) = 2\lambda^2$$

$$S^2 + 2S - 2\lambda^2 - 4\lambda - 2\lambda^2 = 0, \quad S^2 + 2S - 4\lambda^2 - 4\lambda = 0$$

$$\Delta = 4 + 16\lambda^2 + 16\lambda = 4(4\lambda^2 + 4\lambda + 1) = 4(2\lambda + 1)^2$$

$$S = \frac{-2 + 2(2\lambda + 1)}{2} = \frac{-2 + 4\lambda + 2}{2} = 2\lambda, \quad P = \lambda^2 + 2\lambda - 2\lambda = \lambda^2$$

$$S = 2\lambda; P = \lambda^2 \Rightarrow a = b = \lambda \Rightarrow \begin{cases} \sqrt{\lambda^2 + x} = \lambda \\ \sqrt{\lambda^2 - x} = \lambda \end{cases} \Rightarrow \begin{cases} \lambda^2 + x = \lambda^2 \\ \lambda^2 - x = \lambda^2 \end{cases} \Rightarrow x = 0$$