

ROMANIAN MATHEMATICAL MAGAZINE

Let $a > 1$ fixed. Solve for reals:

$$a^x + a^{1/x} = 2a$$

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$$a^x + a^{1/x} = 2a \Leftrightarrow a^{x-1} + a^{1/x-1} = 2 \Leftrightarrow e^{(x-1)\ln a} + e^{(1/x-1)\ln a} = 2 \quad (\ln a > 0)$$

Case 1: $x > 0$

$$x + 1/x \geq 2 \Rightarrow (x-1) + (1/x-1) \geq 0$$

$$(x-1)\ln a + (1/x-1)\ln a \geq 0$$

$$e^{(x-1)\ln a} + e^{(1/x-1)\ln a} \geq 2\sqrt{e^{(x+1/x-2)\ln a}} \geq 2\sqrt{e^0} = 2$$

$$e^{(x-1)\ln a} + e^{(1/x-1)\ln a} = 2 \Leftrightarrow x = 1/x \Rightarrow x = 1$$

Case 2: $x < 0$

$$x < 0 \Rightarrow x-1 < -1 \Rightarrow (x-1)\ln a < -\ln a$$

$$1/x < 0 \Rightarrow 1/x-1 < -1 \Rightarrow (1/x-1)\ln a < -\ln a$$

$$e^{(x-1)\ln a} < e^{-\ln a} = \frac{1}{a}, \quad e^{(1/x-1)\ln a} < e^{-\ln a} = \frac{1}{a}$$

$$e^{(x-1)\ln a} + e^{(1/x-1)\ln a} < 2/a < 2 \quad (\text{since } a > 1)$$

$$\Rightarrow e^{(x-1)\ln a} + e^{(1/x-1)\ln a} \neq 2 \Rightarrow x \neq 1$$