

ROMANIAN MATHEMATICAL MAGAZINE

If $a > 1$ fixed then solve for real numbers:

$$\sqrt{a^x - 1} + \sqrt{a^x + 1} = \sqrt{2}$$

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Domain $a^x - 1 \geq 0$ or, $a^x \geq 1$ or, $x \geq 0$ as $a > 1$

$$\text{Let } y = a^x \text{ then } \sqrt{a^x - 1} + \sqrt{a^x + 1} = \sqrt{2}$$
$$\sqrt{y - 1} + \sqrt{y + 1} = \sqrt{2} \text{ or } (\sqrt{y - 1} + \sqrt{y + 1})^2 = (\sqrt{2})^2$$

$$y + 1 + y - 1 + 2\sqrt{y^2 - 1} = 2$$

$$y + \sqrt{y^2 - 1} = 1 \text{ or } \sqrt{y^2 - 1} = 1 - y$$

$$\text{L.H.S} \geq 0 \Rightarrow \text{R.H.S} \geq 0 \text{ or } 1 - y \geq 0 \text{ or } y \leq 1$$

But $y = a^x$ ($a > 1$) ≥ 1 so we can say $y = 1$ or $a^x = 1$ or $x = 0$ (as $a > 1$).