ROMANIAN MATHEMATICAL MAGAZINE

If $f:\mathbb{R}\to\mathbb{R},\ f(f(x))+f(x)=-x, \forall x\in\mathbb{R}$ then find:

$$\Omega=\int_0^{\pi} f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)dx$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\begin{cases} f(f(x))+f(x)=-x \\ f(f(x))=-x-f(x)\end{cases}$$

$$f\left(f(f(x))\right)+f(f(x))=-f(x),\qquad f\left(f(f(x))\right)+(-x-f(x))=-f(x)$$

$$f\left(f(f(x))\right)-x-f(x)+f(x)=0,\ f\left(f(f(x))\right)-x=0,\qquad f^3(x)=x$$

$$\textit{Identity function}\ :\ x\overset{f}{\rightleftarrows}f(x)\overset{f}{\rightleftarrows}f(f(x))\overset{f}{\rightleftarrows}f\left(f(f(x))\right)\overset{f}{\rightleftarrows}x$$

$$\Omega=\int_0^{\pi} f\left(f\left(f\left(\frac{1}{2+\cos x}\right)\right)\right)dx=\int_0^{\pi}\frac{1}{2+\cos x}dx\overset{x=\pi-y}{\cong}$$

$$=\int_{-\pi}^{\pi}\frac{1}{2+\cos(\pi-y)}dy=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{-\pi-\varepsilon}^{\pi+\varepsilon}\frac{1}{2+\cos(\pi-y)}dy$$

$$=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{-\pi-\varepsilon}^{\pi+\varepsilon}\frac{1}{2-\cos y}dy\overset{y=2\arctan z}{\cong}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{1}{2-\frac{1-z^2}{1+z^2}}\cdot\frac{2}{1+z^2}dz=$$

$$=\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{2}{2+2z^2-1+z^2}dz=2\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{1}{1+3z^2}dz=$$

$$=\frac{2}{3}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_{\tan\frac{-\pi-\varepsilon}{2}}^{\tan\frac{\pi-\varepsilon}{2}}\frac{1}{\frac{1}{3}+z^2}dz=$$

$$=\frac{2\sqrt{3}}{3}\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(arctan\left(\sqrt{3}\cdot tan\frac{\pi-\varepsilon}{2}\right)-arctan\left(\sqrt{3}\cdot tan\frac{\pi-\varepsilon}{2}\right)\right)=$$

$$=\frac{2\sqrt{3}}{3}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{2\pi\sqrt{3}}{3}$$