ROMANIAN MATHEMATICAL MAGAZINE

Find all positive integers k such that $(k+2)^2+3^k$ is a perfect square.

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Solution by Jenish Rijal-Nepal

$$\text{Let } (k+2)^2+3^k = n^2 \qquad (\text{where } n \in \mathbb{N})$$

$$\Rightarrow n^2-(k+2)^2=3^k \Rightarrow (n+k+2)(n-k-2)=3^k$$

$$\text{We know that since both } (n+k+2) \text{ and } (n-k-2)$$
$$\text{are powers of } 3, \text{and are distinct.}$$

$$\text{Therefore they differ by at least a factor of } 3.$$

$$\Rightarrow n+k+2 \geq 3(n-k-2) \Rightarrow 3n-n \leq k+2+3(k+2) \Rightarrow n \leq 2(k+2)$$
$$\text{Now}: (k+2)^2+3^k = n^2 \leq [2(k+2)]^2$$
$$\Rightarrow 3^k \leq 3(k+2)^2 \Rightarrow k \leq 4$$
$$\text{and only } k=2 \text{ works.}$$

$$\therefore k=2 \text{ is the only solution!}$$