ROMANIAN MATHEMATICAL MAGAZINE

Solve for integers:

$$x^3+y^3+5(x+y)=3x^2+2y^2+5xy$$

Proposed by Sakthi Vel-India

Solution by Tapas Das-India

$$x^3+y^3+5(x+y)=3x^2+2y^2+5xy$$
$$(x+y)(x^2-xy+y^2)+5(x+y)=(x+y)(3x+2y)$$
$$(x+y)(x^2-xy+y^2+5-3x-2y)=0$$

$Case\ 1: x+y=0\ then\ x=-y,$

$$Solution\ (x,y)=(t,-t)\ for\ any\ integer\ t$$

$Case\ 2: (x^2-xy+y^2+5-3x-2y)=0$

$$x^2-x(y+3)+(y^2-2y+5)=0$$
$$x=\frac{y+3\pm\sqrt{(y+3)^2-4(y^2-2y+5)}}{2}\quad(1)$$

The $discriminant$:

$$D=(y+3)^2-4(y^2-2y+5)=y^2+6y+9-4(y^2-2y+5)=-3y^2+14y-11$$

$$To\ get\ real\ roots\ D\geq 0\ as\ well\ as\ D\ must\ be\ perfect\ square$$
$$as\ we\ are\ find\ integer\ solutions.$$
$$For\ this\ \ -3y^2+14y-11\geq 0\ or, 3y^2-14y+11\leq 0$$
$$or\ (y-1)(3y-11)\leq 0\ or\ 1\leq y\leq\frac{11}{3}=3.66\ldots$$
$$So\ integer\ values\ of\ y=1,2,3$$
$$y=1\ gives\ from\ (1):\ x=\frac{1+3}{2}=2\ ,so\ (x,y)=(2,1)$$
$$y=3\ \ gives\ from\ (1):\ x=\frac{(3+3)\pm 2}{2}=4,2$$
$$so\ (x,y)=(2,3),(4,3)$$

$$For\ y=2\ ,the\ discriminant\ D=5\ not\ a\ perfect\ square\ number\ ,so\ y\neq 2$$

$$Required\ solutions\ (x,y)=(2,1),(2,3),(4,3),(t,-t),t\ is\ an\ integer.$$