ROMANIAN MATHEMATICAL MAGAZINE

Solve for integers:

$$x^3+2y^3+3x+4y=4x^2+4y^2$$

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$$x^3+2y^3+3x+4y=4x^2+4y^2$$
$$\Rightarrow x(x^2-4x+3)+2y(y^2-2y+2)=0$$
$$\Rightarrow x(x-1)(x-3)+2y((y-1)^2+1)\ =0\ (1)$$
$$\textit{Clearly sign of entire y term } 2y((y-1)^2+1)$$
$$\textit{is depend on y as } (y-1)^2+1\ge 1$$

$\textit{Case } 1: y=0 \textit{ then } x(x-1)(x-3)=0\quad \Rightarrow x=0,1,2$

$$\textit{so solutions are } (x,y)=(0,0),(1,0),(3,0)$$

$\textit{Case } 2: y>0 \textit{ then } \ 2y((y-1)^2+1)>0 \textit{ and from } (1)$

$$x(x-1)(x-3) \textit{must be } <0 \textit{ so } x<0 \textit{ or}, 1<x<3$$

$\textit{Sub case } A: 1<x<3 \textit{ then only integer } x=2 \textit{ then from}(1)\textit{we get}$

$$2(2-1)(2-3)+2y((y-1)^2+1)=0 \textit{ or}, 2y^3-4y^2+4y-2=0$$
$$y^3-2y^2+2y-1=0 \textit{ or}, (y-1)(y^2-y+1)=0 \textit{ or}, y=1$$
$$\textit{as } \ y^2-y+1=0 \textit{ does not have real root since}$$
$$D=1-4=-3<0\ , \textit{solution } (x,y)=(2,1)$$
$$\textit{If } y>1 \textit{ so } y\ge 2(\textit{Integer}), \textit{for } y=2 \textit{ and } x=2\ \ \textit{from}(1)\textit{we get}$$
$$-2+2.2(1^2+1)=-2+8\neq 0\ , \textit{not satisfied}\ ,$$
$$\textit{so their are not integer solution for } y>1$$

$\textit{Sub case } B: x<0\ , \textit{if } y=2\ \ \textit{from}(1)\textit{we get}$

$$x(x-1)(x-3)+2.2((2-1)^2+1)=0 \textit{ or}, x(x-1)(x-3)+8=0$$
$$\textit{Clearly } x=-1 \textit{ satisfy above relation}\ ,$$
$$\textit{absolute value of } x(x-1)(x-3)\textit{increases as x becomes more negative}$$
$$\textit{so only solution } (x,y)=(-1,2)$$

$$\textit{If } y=3 \textit{ then from}(1)\textit{we get } x(x-1)(x-3)+2.3((3-1)^2+1)=0$$
$$\textit{or } x(x-1)(x-3)+30=0$$

$$\textit{Clearly } \ x=-2 \textit{ satisfy above relation}\ , \textit{so solution } (x,y)=(-2,3)$$

$$\textit{If } y=4\quad \textit{from}(1)\textit{we get}:$$
$$x(x-1)(x-3)+2.4((4-1)^2+1)=0\ \ \textit{or}, x(x-1)(x-3)+80=0$$
$$\textit{If we take } x=-3 \textit{ then } x(x-1)(x-3)=-72\neq -80$$
$$\textit{If we take } x=-4 \textit{ then } x(x-1)(x-3)=-140\neq -80$$

$so\ no\ solution\ exist\ since$
$|x(x-1)(x-3)|\ is\ increases\ as\ x\ becomes\ more\ negative$

$Case\ 3:\ y<0\ ,so\ 2y((y-1)^2+1)>0\ ,from\ (1)we\ have$
$x(x-1)(x-3)>0\ or, 0<x<1\ or, x>3$

$We\ must\ consider\ x\geq 4\ as\ no\ integer\ in\ (0,1)$
$sub\ case\ A\ , x\geq 4\ and\ y=-1$
$from\ (1)we\ have\ x(x-1)(x-3)-10=0$
$for\ \ x=4\ , x(x-1)(x-3)=12\neq 10\ and\ value\ of x(x-1)(x-3)\ increases\ for$
$x\geq 4\ ,so\ no\ integer\ solution\ exist$

$If\ y=-2\ \ then\ \ from\ (1)we\ have$
$x(x-1)(x-3)-40=0$
$Clearly\ x=5\ satisfy\ above\ expression\ \ and\ \ value\ of$
$x(x-1)(x-3)\ increases\ for\ x\geq 4$
$so\ \ only\ solution\ (x,y)=(5,-2)$
$Combine\ all\ above\ result\ required\ solutions\ are$
$(x,y)=(0,0),(1,0),(3,0),(-1,2),(-2,3),(5,-2),(2,1)$