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$\textit{Find all positive integers such that:}$

$$6^x + 2^y + 2 = z^2$$

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$\textit{Here, if both } x, y \textit{ are } > 1, \textit{ then the given equation becomes:}$

$$z^2 = 6^x + 2^y + 2 \equiv 0 + 0 + 2 \equiv 2 \pmod 4$$

$\textit{which is a contradiction because } 2 \textit{ is not a quadratic residue of mod } 4.$

$\textit{Case I: } x = 1,$

$6^1 + 2^y + 2 = z^2 \Rightarrow 2^y + 8 = z^2 \textit{ if } y \geq 4 \textit{ then } z^2 = 2^y + 8 \equiv 0 + 8 \equiv 8 \pmod{16}$

$\textit{which is also a contradiction because } 8 \textit{ is not a quadratic residue of mod } 16.$

$\therefore y < 4.$ $\quad$ $\textit{Testing gives } (x, y, z) = (1, 3, 4) \textit{ as the only solution of this case.}$

$\textit{Case II: } y = 1,$

$6^x + 2^1 + 2 = z^2 \Rightarrow z^2 = 6^x + 4$

$\Rightarrow z^2 = 6^x + 4 \equiv (-1)^x + 4 \equiv 1 + 4, -1 + 4 \equiv 3, 5 \pmod 7$

$\textit{which is again a contradiction because } 3, 5 \textit{ are not quadratic residues of mod } 7.$

$\Rightarrow \textit{No solution from this case.}$

$\textit{Therefore, the only solution to the given equation is } (x, y, z) = (1, 3, 4).$